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Convergence to Stationary States for Infinite Harmonic Systems

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We study the evolution of the states for one-dimensional infinite harmonic systems, interacting through a translation invariant force of rapid decrease. We prove that for a large class of initial states convergence to a Gaussian limiting state, as time goes to infinity, is equivalent to convergence of the covariance. The main assumption on the initial states is a kind of weak dependence between distant regions (mixing condition). We prove also convergence of the covariance under some general assumptions. We show furthermore that there are two countable families of intensive constants of the motion, which are "inherited" from the corresponding finite systems. The translation invariant limiting states are in one-to-one correspondence with the admissible values of these constants of the motion. Moreover, under some additional regularity assumption, such states are shown to be Gibbs states, obtained by a "Boltzmann–Gibbs" prescription.

KEY WORDS: Harmonic oscillators; convergence to equilibrium; first integrals.

1. INTRODUCTION

In statistical mechanics it proved to be very fruitful to study infinite-particle systems as idealized models of real (i.e., large, finite) systems. "Thermodynamics" holds in fact only approximately for large finite systems. This approach has been particularly successful in equilbrium statistical mechanics where it led to the theory of Gibbs states.

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Nonequilibrium behavior, however, is still poorly understood. In particular we do not know under what conditions the Boltzmann–Gibbs (BG) postulate on convergence to equilibrium, on which statistical mechanics is based, holds. According to the BG postulate we expect the probabilistic features of an infinite autonomous system to be asymptotically described for large times by a "Gibbs equilbrium distribution," i.e., by a Gibbs state determined by the interaction potential and, in general, by three additional parameters: temperature, chemical potential, and average momentum. These parameters are connected with the three "classical" constants of the motion, i.e., total energy, particle number, and total momentum (it is believed that for physically realistic potentials there are no other additive global constants of the motion). In order to give a rigorous foundation to the BG postulate one should first construct the time evolution for a set of points of the infinite particle phase space which is large enough to contain the support of a large class of initial states ("nonequilibrium dynamics"), and then prove that, under certain conditions, the evoluted states converge weakly to equilibrium (see Ref. 1 for a discussion of the BG postulate for infinite systems). The particular case in which the initial state is absolutely continuous with respect to a given equilibrium state can be reduced to the study of the ergodic properties of the "equilibrium dynamical system," which is defined by the evolution of the infinite system on a set of full measure with respect to the equilibrium Gibbs state (equilibrium dynamics). Physically this corresponds to studying the relaxation of local perturbations. For realistic potentials even the latter task is at present very hard, although equilibrium dynamics has been constructed under fairly general assumptions. On the other hand, in nonequilibrium dynamics there are good results only in dimension one and two for continuous systems (see Ref. 2), or in special cases for lattice systems (see Ref. 3).

In such a situation it was natural to consider some idealized models of particle interaction which are tractable and can teach us something on nonequilibrium behavior. Such models are essentially the free gas, the elastic hard rods in dimension one, and the harmonic oscillators. The equilibrium dynamical system for the free gas and the one-dimensional hard rods were studied respectively in Refs. 4 and 5. For the free gas convergence to equilibrium for initial states which are in general singular with respect to the equilibrium states has been sketched in Ref. 1 and proved in a much more general case in Ref. 6. For the one-dimensional hard rods convergence to equilibrium for a large class of initial translational invariant states has been proved in Ref. 6. In both cases the class of the limit Gibbs states depends on an infinite number of additional parameters (a functional parameter) due to the fact that there is an infinite number of additional constants of the motion (the set of the initial velocities is preserved).

It is worth remembering that in the huge physical literature devoted to harmonic oscillators, actually infinite systems play a considerable role, starting with Hamilton, who considered the evolution of an infinite one-dimensional chain with nearest neighbor interactions (see for instance Refs. 7 and 10). In particular, convergence to a limit (Gaussian) distribution for this model has been examined by Klein and Prigogine.[8] For what concerns rigorous results nonequilibrium dynamics has been constructed under general assumptions by Lanford and Lebowitz[9] and by van Hemmen,[10] who also studied the ergodic properties of the equilibrium dynamical system.

In particular it was found that, if the force matrix $\mathcal{V}$ has an absolutely continuous spectrum, the equilibrium dynamical system is (Bernoulli and hence) mixing, which implies relaxation to equilibrium of local perturbations.

In 1977 Lebowitz and Spohn (see Ref. 11) obtained a real nonequilibrium result for an infinite one-dimensional harmonic chain. They obtained convergence to a stationary state, starting from an initial state which is described far away to the left and to the right by equilibrium states with different temperatures. In the limiting stationary state, contrary to what is expected for a physically reasonable interaction, there is a steady heat flux.

The purpose of this work is to investigate the asymptotic behavior of infinite harmonic systems of identical oscillators with translational invariant interactions in a general framework. We consider one-dimensional systems for simplicity, although there are no essential restrictions, in principal, on the lattice dimensions. We assume also fast (exponential) decrease of the interaction strength as a function of the distance between oscillators. We give now a short description of our results.

We prove a convergence theorem in two steps, first proving that for a large class of initial states convergence to a stationary Gaussian state is equivalent to convergence of the covariance (Section 3), and then giving sufficient conditions for convergence of the covariance (Section 4). In the first step a crucial assumption is a condition of weak dependence between distant regions in the initial state (mixing condition). Since the oscillator dynamical variables (i.e., q_i and p_i) at time t are written as a linear combination of the initial positions and velocities with uniformly small (for large t) coefficients, the proof that convergence of the covariance implies weak convergence of the states appears as a limit theorem for sums of weakly dependent random variables (r.v.'s). The situation recalls the case of free fermions, in the work of Lanford and Robinson.[12] Moreover, linearity implies that conditions for convergence of the covariance can be given in terms of the initial covariance only. We consider the situation in which the initial covariance far away to the left and to the right is near to some translation invariant, in general different, limiting covariances (which generalizes the case of Lebowitz and Spohn), and the case of spatially periodic initial covariance.

In the last section (Section 5) we prove that there are two countable families of "intensive" constants of the motion, which are defined almost everywhere and are almost sure functions with respect to a large class of states (among the assumptions on the states a crucial role is again played by a mixing condition). In physical terms the elements of the first set are the Fourier components of the energy density of the "normal modes" in the Brillouin zone $[-\pi,\pi]$; the elements of the second family are connected with the degeneracy of the characteristic frequencies which appears already in the corresponding finite system.

It is interesting to remark that even if these constants of the motion are almost sure functions, the values which they assume in the initial state do not necessarily coincide with their values in the limiting Gaussian state [however, they do coincide if the initial state is spatially homogeneous (translation invariant)]. This is because, by considering weak convergence, we look at the "local" behavior of the system as $t \to \infty$. (The specific entropy of the free gas behaves in a similar way, see Ref. 1.)

Finally, in the Appendix we give the proofs of some estimates on oscillating integrals which are used in the text.

2. NOTATIONS AND PRELIMINARY RESULTS

Throughout this paper we denote by $\mathbb{Z}_+ = \{k \in \mathbb{Z} \mid k > 0\}$ and by $\mathbb{R}_+ = \{x \in \mathbb{R}^1 \mid x > 0\}$ the positive integer and real numbers, respectively. By $\overline{\mathbb{Z}}_+ = \mathbb{Z}_+ \cup \{0\}$ and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{0\}$ we denote the corresponding non-negative sets. The Euclidean norm in $\mathbb{R}^n$ is denoted by $|\cdot|$, the L_p norm of a random variable ξ ($p \geqslant 1$, not necessarily integer), by $\|\xi\|_p$. The Fourier (anti)transform of a square summable sequence $\{f_k \in \mathbb{C}\}_{k\in\mathbb{Z}}$ is denoted by $\hat{f}(\theta)$, $\theta \in [-\pi,\pi] : \hat{f}(\theta) = \sum_{k\in\mathbb{Z}} f_k \exp(ik\theta)$. Finally, as usual, we denote by $(\cdot,\cdot)$ the scalar product in a Euclidean space, and by $(\ )^T$ matrix transposition.

Consider an infinite system of oscillators on the line $\mathbb{R}^1$, such that the kth oscillator, $k \in \mathbb{Z}$, has its equilibrium position at k. We denote by $q_k \in \mathbb{R}^1$, $p_k \in \mathbb{R}^1$, respectively, the displacement from the equilibrium position and the momentum of the kth oscillator. By $x_k \in \mathbb{R}^2$, $k \in \mathbb{Z}$ we denote the vector with components $x_k^{(1)} = q_k$, $x_k^{(2)} = p_k$.

Definition 1.1. The phase space of the infinite oscillator system is the set $\mathcal{X}$ of all the sequences $x = \{x_k\}_{k\in\mathbb{Z}}$ with $x_k \in \mathbb{R}^2$.

Endowed with the natural topology of coordinatewise convergence, $\mathcal{X}$ is a polish space. If $N \subseteq \mathbb{Z}$ is a subset of the integers we denote by $\mathfrak{B}_N$ the smallest σ algebra of subsets of $\mathfrak{X}$ with respect to which the coordinates $x_k^{(\alpha)}$, $k \in N$, $\alpha = 1,2$ are measurable. If $N = \{k \in \mathbb{Z} \mid k \geqslant h\}$ $(N = \{k \in \mathbb{Z} \mid k \leqslant h\})$, $h \in \mathbb{Z}$, we shall write $\mathfrak{B}_h^{+\infty}$ $(\mathfrak{B}_{-\infty}^{n})$ instead of $\mathfrak{B}_N$. The σ algebra $\mathfrak{B}_{\mathbb{Z}}$ is denoted simply by $\mathfrak{B}$. $\mathfrak{B}$ coincides with the Borel σ algebra of $\mathfrak{X}$ with respect to the natural topology.[9]

Definition 1.2. By a state P we mean a probability measure on the measurable space $(\mathfrak{X}, \mathfrak{B})$.

If the oscillators interact with a quadratic potential (harmonic approximation) and they have all the same mass (equal to one) the equations of the motion can be written formally as

$$\begin{aligned} \dot{q}_k(t) &= p_k(t) \\ \dot{p}_k(t) &= -\sum_{h \in \mathbb{Z}} \mathcal{V}_{k,h}\, q_h(t) \end{aligned} \tag{2.1}$$

where $\mathcal{V} = \{\mathcal{V}_{k,h}\}$ $k \in \mathbb{Z}$, $h \in \mathbb{Z}$ is a symmetric matrix called the force matrix. We can write Eq. (2.1) as a differential equation in $\mathfrak{X}$:

$$\frac{d}{dt} x(t) = Ax(t) \tag{2.1$'$}$$

where $(Ax)_k = (p_k, -\sum_h \mathcal{V}_{k,h} q_h)$. The problem of dynamics consists in finding a solution of Eq. (2.1′) for a given initial condition

$$x(0) = x_0 \tag{2.2}$$

Before discussing this problem we state our assumptions on the force matrix.

Assumption I. $\mathcal{V}$ is symmetric and translation invariant, i.e., $\mathcal{V}_{k,h} = V_{k-h}$ for some even sequence $\{V_k\}_{k \in \mathbb{Z}}$.

Assumption II. There are two positive constants c_1, c_2 such that for all $k \in \mathbb{Z}$ the following inequality holds:

$$|V_k| \leqslant c_1 \exp(-c_2 |k|)$$

Assumption III. The "dynamical function" $\omega^2(\theta) = \hat{V}(\theta) = \sum_{k \in \mathbb{Z}} V_k e^{ik\theta}$ is bounded away from 0, i.e., $\min_{\theta \in [-\pi,\pi]} \omega^2(\theta) > 0$.

Note that Assumption II implies that $\omega^2(\theta)$ is an analytic periodic function of θ. Assumption III is equivalent to the requirement that the spectrum of $\mathcal{V}$ as an operator on $l_2(\mathbb{Z})$ is positive and bounded away from 0. This assumption is important in the one-dimensional case, since otherwise we would get into difficulties connected to the fact that the equilibrium state does not exist in the ordinary sense (see Ref. 9). Assumption II, on the other hand, is made only for simplicity: exponential decay could be replaced by a sufficiently rapid power decay.

The existence of dynamics is stated by the following theorem which is proved in Refs. 9 and 10.

Theorem 2.1. Let $\mathfrak{X}' = \{x \in \mathfrak{X} \mid \sup_{k \in \mathbb{Z}}(|x_k|^2/(k^2+1)^m) < +\infty$ for some $m \in \mathbb{Z}_+\}$. Then if the force matrix $\mathcal{V}$ satisfies Assumptions I and II and $x_0 \in \mathfrak{X}'$ there is a unique solution $x(t)$ of the initial data problem (2.1′)–(2.2) such that $x(t) \in \mathfrak{X}'$ for all $t \in \mathbb{R}'$.

The solution is linear in the initial data:

$$x(t) = \mathcal{U}_t x_0 \tag{2.3}$$

the evolution matrix $\mathcal{U}_t$ being given by

$$(\mathcal{U}_t x)_k = \sum_{h \in \mathbb{Z}} \mathcal{U}_{k-h}(t) x_h, \qquad x \in \mathfrak{X}' \tag{2.4}$$

and $\mathcal{U}_k(t)$, $k \in \mathbb{Z}$, $t \in \mathbb{R}'$ are the Fourier coefficients of the function matrix

$$\hat{\mathcal{U}}(\theta, t) = \begin{bmatrix} \cos[\omega(\theta)t] & \dfrac{\sin[\omega(\theta)t]}{\omega(\theta)} \\ -\omega(\theta)\sin[\omega(\theta)t] & \cos[\omega(\theta)t] \end{bmatrix} \tag{2.5}$$

[Note that $\hat{\mathcal{U}}$ is analytic in θ and therefore the series in Eq. (2.4) converges absolutely for $x \in \mathfrak{X}'$, and $x(t) \in \mathfrak{X}'$ for all $t \in \mathbb{R}^1$.]

We have now a one-parameter group of transformations of $\mathfrak{X}'$ into itself in terms of which we can define the evolution of initial states.

Definition 2.3. If P is a state such that $P(\mathfrak{X}') = 1$ we shall call evolution of the state P for the harmonic dynamics with matrix force $\mathcal{V}$ the family of states $\{P_t,\ t \in \mathbb{R}^1\}$ given by

$$P_t(A) = P(\mathcal{U}_{-t}(A \cap \mathfrak{X}')), \qquad A \in \mathfrak{B}, \quad t \in \mathbb{R}^1$$

where $\mathcal{U}_t$, $t \in \mathbb{R}^1$ is given in terms of $\mathcal{V}$ by Eq. (2.4).

We conclude this paragraph by giving some results of the theory of stochastic processes.

Let P be a state. The function $\alpha_P : \mathbb{Z}_+ \to \overline{\mathbb{R}}_+$ given by

$$\alpha_P(h) = \sup_{k \in \mathbb{Z}} \sup_{\substack{A \in \mathfrak{B}_{h+k}^{+\infty} \\ B \in \mathfrak{B}_{-\infty}^{k}}} |P(A \cap B) - P(A)P(B)|, \qquad h \in \mathbb{Z}_+$$

is called the strong mixing coefficient of P. If $\lim_{h \to +\infty} \alpha_P(h) = 0$ the state P is said to be strong mixing. The following result can be proved by an easy adaptation of the proof of Theorem 17.2.1 of Ref. 13.

Proposition 2.1. If the random variables ξ, η are measurable with respect to the σ algebras $\mathfrak{B}_{-\infty}^{k}$ and $\mathfrak{B}_{h+k}^{+\infty}$, $k \in \mathbb{Z}$, $h \in \mathbb{Z}_{+}$, respectively, and moreover $|\xi| \leqslant N$, $|\eta| \leqslant M$ P-almost everywhere, the following inequality holds:

$$|\mathbb{E}_{P}\xi\eta - \mathbb{E}_{P}\xi\mathbb{E}_{P}\eta| \leqslant 4N \cdot M\alpha_{P}(h) \tag{2.6}$$

(If ξ and η are complex functions 4 is replaced by 16.)

For unbounded random variables we need the following result, which is a modification of Theorem 17.2.2 of Ref. 13.

Proposition 2.2. If the random variables ξ, η are measurable with respect to the σ algebras $\mathfrak{B}_{-\infty}^{k}$ and $\mathfrak{B}_{h+k}^{+\infty}$, respectively, $k \in \mathbb{Z}$, $h \in \mathbb{Z}_{+}$, and moreover $\xi \in L_{r}(\mathfrak{X}, P)$, $\eta \in L_{s}(\mathfrak{X}, P)$, with $r, s \in \mathbb{R}_{+}$ such that $g = 1 - r^{-1} - s^{-1} > 0$, the following inequality holds:

$$|\mathbb{E}_{P}\xi\eta - \mathbb{E}_{P}\xi\mathbb{E}_{P}\eta| \leqslant 10\|\xi\|_{r}\|\eta\|_{s}(\alpha_{P}(h))^{g} \tag{2.7}$$

Proof. Set $\xi_1 = \xi/\|\xi\|_r$, $\eta_1 = \eta/\|\eta\|_s$. If ζ is a r.v., we define the truncated variable $\zeta^{(N)}$, $N \in \mathbb{R}_{+}$ by setting

$$\zeta^{(N)} = \begin{cases} \zeta & \text{if} \quad |\zeta| \leqslant N \\ 0 & \text{if} \quad |\zeta| > N \end{cases}$$

Setting $\xi_1 = \xi_1^{(N)} + \bar{\xi}_1^{(N)}$, $\eta_1 = \eta_1^{(N)} + \bar{\eta}_1^{(N)}$ we have by Proposition 2.1 $|\mathbb{E}_{P}\xi_1^{(N)}\eta_1^{(M)} - \mathbb{E}_{P}\xi_1^{(N)}\mathbb{R}_{P}\eta_1^{(M)}| \leqslant 4NM\alpha_{P}(h)$. Moreover, $\mathbb{E}_{P}|\bar{\xi}_1^{(N)}| \leqslant N^{1-r}\int_{\mathfrak{X}}|\xi_1|^{r}P(dx) = N^{1-r}$, and, similarly, $\mathbb{E}_{P}|\bar{\eta}_1^{(M)}| \leqslant M^{1-s}$. Therefore $\mathbb{E}_{P}|\xi_1^{(N)}\bar{\eta}_1^{(M)}| \leqslant NM^{1-s}$ and $\mathbb{E}_{P}|\bar{\xi}_1^{(N)}\eta_1^{(M)}| \leqslant MN^{1-r}$. Furthermore, choosing $p, q \in \mathbb{R}_{+}$, $p < r$, $q < s$, such that $p^{-1} + q^{-1} = 1$, we have $|\mathbb{E}_{P}(\bar{\xi}_1^{(N)}\bar{\eta}_1^{(M)}) \leqslant N^{1-r/p}M^{1-s/q}$, since $\|\bar{\xi}_1^{(N)}\|_p \leqslant N^{1-r/p}\int_{\mathfrak{X}}|\xi_1|^{p}P(dx) \leqslant N^{1-r/q}$, and similarly, $\|\bar{\eta}_1^{(M)}\|_q \leqslant M^{1-s/q}$. Therefore we find

$$\begin{aligned}
&|\mathbb{E}_{P}(\xi_1\eta_1) - \mathbb{E}_{P}\xi_1\mathbb{E}_{P}\eta_1| \\
&\quad \leqslant |\mathbb{E}_{P}(\xi_1^{(N)}\eta_1^{(M)}) - \mathbb{E}_{P}\xi_1^{(N)}\mathbb{E}_{P}\eta_1^{(m)}| + |\mathbb{E}_{P}(\xi_1^{(N)}\bar{\eta}_1^{(M)}) \\
&\qquad\qquad - \mathbb{E}_{P}\xi_1^{(N)}\mathbb{E}_{P}\bar{\eta}_1^{(M)}| \\
&\qquad + |\mathbb{E}_{P}(\bar{\xi}_1^{(N)}\eta_1^{(M)}) - \mathbb{E}_{P}\bar{\xi}_1^{(N)}\mathbb{E}_{P}\eta_1^{(M)}| + |\mathbb{E}_{P}(\bar{\xi}_1^{(N)}\bar{\eta}_1^{(M)}) - \mathbb{E}_{P}\bar{\xi}_1^{(N)}\mathbb{E}_{P}\bar{\eta}_1^{(M)}| \\
&\quad \leqslant 4NM\alpha_{P}(h) + MN^{1-r} + M^{1-s} + NM^{1-s} \\
&\qquad + N^{1-r} + N^{1-r/p}M^{1-s/q} + N^{1-r}M^{1-s}
\end{aligned}$$

Setting $N = (\alpha_{P}(h))^{-r^{-1}}$, $M = (\alpha_{P}(h))^{-s^{-1}}$ (which implies $N, M \geqslant 1$), it is easily seen that

$$|\mathbb{E}_{P}(\xi_1\eta_1) - \mathbb{E}_{P}\xi_1\mathbb{E}_{P}\eta_1| \leqslant 10(\alpha_{P}(h))^{g}$$

Going back to the original variables ξ, η, we find inequality (2.7). ∎

If P is a state such that $\mathbb{E}_P|x_k|^2 < \infty$ for all $k \in \mathbb{Z}$, we denote by $C_P = \{C_P(h,k)\}_{h,k\in\mathbb{Z}}$ its covariance: $C_P(h,k)$ is a 2×2 matrix with elements

$$C_P^{(\alpha,\beta)}(h,k) = \mathbb{E}_P\big(x_h^{(\alpha)}x_k^{(\beta)}\big) - \mathbb{E}_P x_h^{(\alpha)}\mathbb{E}_P x_h^{(\beta)}, \qquad \alpha,\ \beta = 1,2$$

Let P be a state such that $\mathbb{E}_P|x_k|^2 < \infty$ for all $k \in \mathbb{Z}$. If $\mathbb{E}_P x_k = a$ for all $k \in \mathbb{Z}$ and some fixed $a \in \mathbb{R}^2$, and the covariance C_P is translation invariant, i.e., $C_P(h+1,k+1) = C_P(h,k)$ for all integers h,k, the state P is said to be translation invariant in the wide sense. Translation invariance in the strict sense means invariance of the probabilities of local events with respect to translations. The two notions coincide for Gaussian states. (In probabilistic literature the term "stationary" instead of "translation invariant" is generally used. Here we call "stationary" the states which are invariant with respect to time evolution.) Clearly any positive definite double sequence $C = \{C(h,k)\}_{h,k\in\mathbb{Z}}$, i.e., such that $\sum_{i,j=1}^{n} t_i\bar{t}_j C^{(\alpha_i,\alpha_j)}(k_i,k_j)\ 0$ for any choice of $n \in \mathbb{Z}_+$, of the complex numbers t_i and of the indexes $k_i \in \mathbb{Z}$, $\alpha_i = 1,2$, $i = 1,\ldots,n$, is the covariance of some state. We shall call such a sequence a "covariance" even if there is no reference to an underlying state.

If a state is translation invariant in the wide sense, its covariance can be represented in terms of the spectral measure.[13] More precisely, let $C_P(h,k) = R_P(k-h)$, $h,k \in \mathbb{Z}$ be the covariance of such a state. By the Bochner theorem there are four complex functions of bounded variation $\{F^{(\alpha,\beta)}\}_{\alpha,\beta=1,2}$ defined on $[-\pi,\pi]$, such that $R_P = \hat{F}$, i.e.,

$$R_P^{(\alpha,\beta)}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}F^{(\alpha,\beta)}(dx), \qquad \alpha,\ \beta = 1,2, \quad k \in \mathbb{Z}$$

F is said to be the spectral measure of the state P. If $\Delta \subset [-\pi,\pi]$ is a Borel set, the matrix $F(\Delta) = \{F^{(\alpha,\beta)}(\Delta)\}_{\alpha,\beta=1,2}$ is self-adjoint and positive definite. If the functions $F^{(\alpha,\beta)}$ are absolutely continuous with respect to the Lebesgue measure, with densities $f^{(\alpha,\beta)}$, $\alpha,\ \beta = 1,2$, the matrix $f = \{f^{(\alpha,\beta)}\}_{\alpha,\beta=1,2}$ is called the spectral density of the state P.

We conclude with a result concerning the evolution of the covariance under harmonic dynamics.

Proposition 2.3. Let P be a state such that $\mathbb{E}_P(|x_k|^2) \leqslant K(1+k^2)^n$, for some $n \in \mathbb{Z}_+$, $K \in \mathbb{R}_+$ and all $k \in \mathbb{Z}$, and let $\mathcal{V}$ be a force matrix satisfying Assumptions I and II above. Then the state P_t, $t \in \mathbb{R}^1$, defined by Definition 2.3, is such that $\mathbb{E}_{P_t}(|x_k|^2) \leqslant K(t)(1+k^2)^n$ for some constant $K(t)$ and all $k \in \mathbb{Z}$, and the covariance C_{P_t} is given by

$$C_{P_t}(h,k) = \sum_{l,l'\in\mathbb{Z}} \mathfrak{U}_{h-l}(t)C_P(l,l')\mathfrak{U}_{k-l'}^T(t) \tag{2.8}$$

Proof. It is easy to see that if P satisfies the condition above $P(\mathscr{X}')=1$, and therefore P_t makes sense for all $t \in \mathbb{R}^1$. Furthermore we have

$$\mathbb{E}_{P_t}(|x_k|^2) = \mathbb{E}_P\big(|(\mathfrak{U}_t x)_k|^2\big) = \sum_{h,h' \in \mathbb{Z}} \mathbb{E}_P\big(\mathfrak{U}_{k-h}(t)x_h, \mathfrak{U}_{k-h'}(t)x_{h'}\big)$$

Setting

$$R_k^2(t) = \sum_{\alpha,\beta=1}^{2} \big(\mathfrak{U}_k^{(\alpha,\beta)}\big)^2, \qquad k \in \mathbb{Z}, \quad t \in \mathbb{R}^1 \tag{2.9}$$

it follows from the assumptions on $\mathcal{V}$ that $\{R_k(t)\}_{k \in \mathbb{Z}}$ is a sequence of rapid decrease. Since $|\mathfrak{U}_{k-h}(t)x_h| \leqslant R_{k-h}(t)|x_h|$ we obtain

$$\mathbb{E}_{P_t}(|x_k|^2) \leqslant \sum_{h \in \mathbb{Z}} R_{k-h}(t) \sum_{h' \in \mathbb{Z}} R_{k-h'}(t)\mathbb{E}_P(|x_{h'}|^2) \leqslant K(t)(k^2+1)^n$$

where $K(t)$ is some constant independent of k and the last inequality comes from the fact that the convolution of a sequence which is $O((k^2+1)^n)$ with a sequence of rapid decrease is again $O((k^2+1)^n)$. This implies that the covariance C_{P_t} exists. On the other hand it is easy to see that

$$\begin{aligned} C_{P_t}^{(\alpha,\beta)}(k,h) &= \mathbb{E}_{P_t}\big(x_k^{(\alpha)}x_h^{(\beta)}\big) = \mathbb{E}_P\big((\mathfrak{U}_t x)_k^{(\alpha)}(\mathfrak{U}_t x)_h^{(\beta)}\big) \\ &= \sum_{l,l' \in \mathbb{Z}} \sum_{\gamma,\gamma'=1}^{2} \mathfrak{U}_{k-l}^{(\alpha,\gamma)}(t)\mathfrak{U}_{k-l'}^{(\beta,\gamma')}(t)C_P^{(\gamma,\gamma')}(l,l') \end{aligned}$$

which is the same as Eq. (2.8). ■

We shall use the following notion of covergence for the covariance.

Definition 2.4. Let t_0 be a point of the extended real line, we shall say that the family of covariances $\{C_t,\ t \in \mathbb{R}^1\}$ converges, as $t \to t_0$ to the covariance C whenever

$$\lim_{t \to t_0} C_t^{(\alpha,\beta)}(h,k) = C^{(\alpha,\beta)}(h,k)$$

for all α, $\beta = 1,2$, $h,k \in \mathbb{Z}$.

3. THE CONVERGENCE THEOREM

We shall prove the theorem under an additional technical assumption on the force matrix $\mathcal{V}$ which greatly simplifies the estimates.

Assumption IV. The set of the points $\theta \in [-\pi,\pi]$ for which $\omega''(\theta) = \omega'''(\theta) = 0$ is empty.

Theorem 3.1. Let P be a state such that

(i) $$\mathbb{E}_P x_k = 0 \qquad \text{for all} \quad k \in \mathbb{Z},$$

(ii) $$\sup_{k\in\mathbb{Z}} \mathbb{E}_P|x_k|^{4+\lambda} < \infty \qquad \text{for some} \quad \lambda > 0;$$

(iii) $$\sum_{h\in\mathbb{Z}_+} h\cdot(\alpha_P(h))^{\lambda/4+\lambda} < \infty \qquad \text{if} \quad \lambda \leqslant 8, \quad \text{or}$$

$$\sum_{h\in\mathbb{Z}_+} h^2\alpha_P(h) < \infty \qquad \text{if} \quad \lambda > 8.$$

Then if the force matrix $\mathcal{V}$ satisfies Assumptions I–III of Section 1 and Assumption IV above, the states P_t, $t \in \mathbb{R}^1$, associated to P by Definition 1.3 tend weakly, as $t \to \infty$, to the Gaussian state G with covariance C_G if and only if the covariance C_P converge, as $t \to \infty$ to C_G.

Proof. We shall first prove necessity and then sufficiency. In the course of the proof we denote by $c_1, c_2, \ldots$ different (in general) absolute constants.

Proof of Necessity. Since the covariance is not the expectation of a bounded continuous function, convergence of the covariance does not follow from weak convergence of the states. We shall prove the result by an uniform integrability argument, which is based on the following estimate: there is a constant c_1, such that

$$\mathbb{E}_{P_t}\left(|x_k|^4\right) \leqslant c_1, \qquad k \in \mathbb{Z}, \quad t \in \mathbb{R}^1 \tag{3.1}$$

We have

$$\mathbb{E}_{P_t}\left(|x_k|^4\right) = \mathbb{E}_P\left(|(\mathcal{U}_t x)_k|^4\right) = \sum_{\alpha,\beta=1}^{2} \mathbb{E}_P\left(\left[(\mathcal{U}_t x)_k^{(\alpha)}\right]^2\left[(\mathcal{U}_t x)_k^{(\beta)}\right]^2\right)$$

Fixing α and β and setting $a_h(t) = (\mathcal{U}_{k-h}(t)x_h)^{(\alpha)}$, $b_h(t) = (\mathcal{U}_{k-h}(t)x_h)^{(\beta)}$ we get [see Eq. (2.4)]

$$\begin{aligned}
\mathbb{E}_P\left(\left[(\mathcal{U}_t x)_k^{(\alpha)}\right]^2\left[(\mathcal{U}_t x)_k^{(\beta)}\right]^2\right) &= \sum_h \mathbb{E}_P\left(a_h^2 b_h^2\right) + \sum_{h,l}{}' \mathbb{E}_P\left(a_h^2 b_h b_l + a_h a_l b_l^2\right) \\
&\quad + \sum_{h,l}{}' \mathbb{E}_P\left(a_h^2 b_h^2 + a_h b_h a_l b_l\right) \\
&\quad + \sum_{h,l,m}{}' \mathbb{E}_P\left(a_h^2 b_l b_m + a_l a_h b_l b_m + a_h a_l b_m^2\right) \\
&\quad + \sum_{h,l,m,n}{}' \mathbb{E}_P(a_h a_l b_m b_n)
\end{aligned}$$

where the indices h, l, m, n, run over $\mathbb{Z}$ except that $\sum'$ indicates that summation is over distinct indices. We begin by estimating the four-index term. We consider only the sum for $h < l < m < n$, since the other possible orderings are estimated in the same way. Using Proposition 2.2 and the Hölder inequality we get

$$\begin{aligned}|\mathbb{E}_P(a_h a_l b_m b_n)| &\leqslant 10\|a_h a_l b_m\|_{(4+\lambda)/3}\|b_n\|_{4+\lambda}(\alpha_P(n-m))^{\eta}\\ &\leqslant 10c_2^4 R_h' R_l' R_m' R_n'(\alpha_P(n-m))^{\eta}\end{aligned}$$

where $c_2 = \sup_{k\in\mathbb{Z}}\|x_k\|_{4+\lambda}$, $R_h'(t) = R_{k-h}(t)$, $\eta = \lambda/(4+\lambda)$. Setting $d = \sup_{t\in\mathbb{R}}\sum_{k\in\mathbb{Z}} R_k^2(t)$ ($d < \infty$, from the assumptions on $\mathcal{V}$), $j_1 = l-h$, $j_2 = m-l$, $j_3 = n-m$, using the Schwartz inequality for the sums over j_2 and h we get [notice that if $\lambda > 8$ $\sum_{h\in\mathbb{Z}_+} h^2\alpha_P(h) < \infty$ implies $\sum_{h\in\mathbb{Z}_+} h(\alpha_P(h))^{\eta} < \infty$]

$$\begin{aligned}&\sum_{h\in\mathbb{Z}}\sum_{j_3>0}\sum_{\substack{j_1,j_2\\0<j_1,j_2\leqslant j_3}} |\mathbb{E}_P(a_h a_{h+j_1} b_{h+j_1+j_2} b_{h+j_1+j_2+j_3})|\\ &\qquad\leqslant 10c_2^4 d^2 \sum_{j_3>0} j_3(\alpha_P(j_3))^{\eta} = c_3\end{aligned}$$

In the same way we obtain that

$$\sum_{h\in\mathbb{Z}}\sum_{j_1>0}\sum_{\substack{j_2,j_3\\0<j_2,j_3\leqslant j_1}} |\mathbb{E}_P(a_h a_{h+j_1} b_{h+j_1+j_2} b_{h+j_1+j_2+j_3})| \leqslant c_3$$

If $j_2 > \max(j_1, j_3)$ we use the inequality

$$|\mathbb{E}_P(a_h a_l b_m b_n)| \leqslant |\mathbb{E}_P(a_h a_l)\mathbb{E}_P(b_m b_n)| + 10c_2^4 R_h' R_l' R_m' R_n'(\alpha_p(m-l))^{\eta} \tag{3.2}$$

Using again Proposition 2.2 we find

$$\sum_{h\in\mathbb{Z}}\sum_{j>0} |\mathbb{E}_P(a_h a_{h+j})| \leqslant c_4, \qquad \sum_{h\in\mathbb{Z}}\sum_{j>0} |\mathbb{E}_P(b_h b_{h+j})| \leqslant c_4$$

where

$$c_4 = 10c_2^2 d \sum_{j>0} (\alpha_P(j))^{(2+\lambda)/(4+\lambda)}$$

so that from inequality (3.2) we get

$$\sum_{h\in\mathbb{Z}}\sum_{j_2>0}\sum_{\substack{j_1,j_3\\0<j_1,j_3\leqslant j_2}} |\mathbb{E}_P(a_h a_{h+j_1} b_{h+j_1+j_2} b_{h+j_1+j_2+j_3})| \leqslant c_3 + c_4^2 = c_5$$

Taking into account all the possible index orderings we see that the four-index term is less in absolute value than $4!(2c_3 + c_5)$. The proof that the three- and two-index terms are bounded by an absolute constant is done in a similar way. For the one-index term it is obvious. Inequality (3.1) is proved.

Consider now the new variables

$$\left(x_j^{(K)}\right)^{(\alpha)} = \begin{cases} x_j^{(\alpha)} & \text{if } |x_j^{(\alpha)}| < K \\ x_j^{(\alpha)}\left(2 - x_j^{(\alpha)}/K\right) & \text{if } |x_j^{(\alpha)}| \in [K, 2K) \quad \alpha = 1,2, \quad K \in \mathbb{R}_+ \\ 0 & \text{if } |x_j^{(\alpha)}| \geqslant 2K \end{cases} \tag{3.3}$$

Setting $\bar{x}_j^{(K)} = x_j - x_j^{(K)}$, $j \in \mathbb{Z}$, we have, using inequality (3.1) $\mathbb{E}_{P_t}(|\bar{x}_j^{(K)}|^2) \leqslant \sqrt{c_1}\,/K$, which implies that for all $j, h \in \mathbb{Z}$, α, $\beta = 1,2$, $t \in \mathbb{R}^1$

$$|\mathbb{E}_{P_t}\left(x_j^{(\alpha)} x_h^{(\beta)}\right) - \mathbb{E}_{P_t}\left(\left(x_j^{(K)}\right)^{(\alpha)}\left(x_h^{(K)}\right)^{(\beta)}\right)| \leqslant c_1'/K \tag{3.4}$$

On the other hand weak convergence implies that for all $j, h \in \mathbb{Z}$, α, $\beta = 1,2$

$$\lim_{t\to\infty} \mathbb{E}_{P_t}\left(\left(x_j^{(K)}\right)^{(\alpha)}\left(x_h^{(K)}\right)^{(\beta)}\right) = \mathbb{E}_G\left(\left(x_j^{(K)}\right)^{(\alpha)}\left(x_h^{(K)}\right)^{(\beta)}\right) \tag{3.5}$$

since $x_k^{(K)}, k \in \mathbb{Z}$, are continuous bounded functions on $\mathcal{X}$. By standard arguments it is easily seen that inequality (3.4) and equality (3.5) imply convergence of the covariance. Necessity is proved.

Proof of Sufficiency. We prove first convergence to a Gaussian distribution for the single variables $x_k^{(\alpha)}(t) = (\mathcal{U}_t x)_k^{(\alpha)}$, $k \in \mathbb{Z}$, $\alpha = 1,2$. Without loss of generality we can consider the variable $x_0^{(1)}(t) = q_0(t)$ and assume that $t \to +\infty$. We assume that the limit dispersion $\sigma^2 = \lim_{t\to+\infty} \mathbb{E}_{P_t}(q_0^2) = \lim_{t\to+\infty} \mathbb{E}_P([q_0(t)]^2) = c_G^{(1,1)}(0,0)$ is positive, since if $\sigma^2 = 0$ the limit variable is obviously (degenerate) Gaussian. $q_0(t)$ is a linear combination of the r.v.'s x_k, $k \in \mathbb{Z}: q_0(t) = \sum_{k\in\mathbb{Z}} (\mathcal{U}_k(t) x_k)^{(1)}$. As a first step we replace the r.v.'s x_k by the truncated variables $y_k^{(K)} = x_k^{(K)} - \mathbb{E}_P x_k^{(K)}$, $k \in \mathbb{Z}$, where $x_k^{(K)}$, $K \in \mathbb{R}_+$, is defined by Eq. (3.3), and prove a central limit theorem for the r.v. $\xi(t) = \sum_{k\in\mathbb{Z}} (\mathcal{U}_{-k}(t) y_k^{(K)})^{(1)}$. The proof is based on a variant of a standard technique of Bernstein for sums of weakly dependent r.v.'s. The inequality

$$R_l(t) \leqslant c/(1 + |t|^{1/3}), \qquad l \in \mathbb{Z} \tag{3.6}$$

which follows by Proposition A.3 (Appendix) from the assumptions on the force matric $\mathcal{V}$, shows that $\xi(t)$ is a sum of "uniformly small" r.v.'s, and plays an important role in the proof.

Let $\beta, \delta: \mathbb{R}_+ \to \mathbb{Z}_+$ be two nondecreasing functions such that $\lim_{t\to+\infty} \beta(t) = \lim_{t\to+\infty} \delta(t) = +\infty$ and moreover that (a) $\beta(t) = o(t^{1/2})$, (b) $t^{1/3}\delta(t)/\beta(t) = o(1)$. Two such functions can be constructed in the following way:

Let $s: \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that $\lim_{x\to+\infty} s(x) = +\infty$ and set $f(x) = x^6 s(x)$, $\bar{\delta} = f^{-1}$, where f^{-1} is the inverse function of

f. Set furthermore $\bar{\beta}(x) = \bar{\delta}^3(x)[s(\bar{\delta}(x))]^b$ for some $b \in (1/3, 1, 2)$, and $g(x) = x^3\alpha_P([x])$. Since $x = \bar{\delta}^6(x)s(\bar{\delta}(x))$ we have $\bar{\beta}(t) = t^{1/2}[s(\bar{\delta}(t))]^{-1/2+b}$, $\bar{\delta}(t)t^{1/3}/\bar{\beta}(t) = [s(\bar{\sigma}(t))]^{1/3-b}$ and $[t/\bar{\beta}(t)]\alpha_P([\bar{\delta}(t)])$ $= [s(\bar{\delta}(t))]^{1-b}g(\bar{\delta}(t))$. From Assumption (iii) of our theorem it follows that $\lim_{x\to\infty} g(x) = 0$. [This is easily seen if $\lambda > 8$. For $\lambda \leqslant 8$ it is proved in the same way since then $\sum_{h\in\mathbb{Z}_+} h(\alpha_P(h))^{\lambda/4+\lambda} < \infty$ implies $\sum_{h\in\mathbb{Z}} h^2\alpha_P(h)$ $< \infty$.] Therefore if we take as s a nondecreasing function diverging at $+\infty$ and such that $s = o(g^{-1/1-b})$ it is easily seen that the functions $\beta(t)$ $= [\bar{\beta}(t)]$ and $\delta(t) = [\bar{\delta}(t)]$ have the required properties.

Setting $a_k(t) = (\mathcal{U}_{-k}(t)y_k^{(K)})^{(1)} = (\mathcal{U}_k(t)y_k^{(K)})^{(1)}$ we have $\xi(t) = \sum_{k\in\mathbb{Z}} a_k(t)$. The dispersion $D\xi(t)$ of $\xi(t)$ is bounded uniformly in t (and in K): in fact using Proposition 2.2 we have

$$D\xi(t) = \mathbb{E}_P\left(\left[\xi(t)\right]^2\right) = \sum_{l\in\mathbb{Z}} \mathbb{E}_P\left(\left[a_k(t)\right]^2\right) + \sum_{l,h\in\mathbb{Z}}{}' \mathbb{E}_P\left(a_l(t)a_h(t)\right)$$
$$\leqslant c_2^2 d\left(1 + 20d\sum_{j>0}\left[\alpha_P(j)\right]^{(2+\lambda)/(4+\lambda)}\right) = c_6$$

Moreover, as we shall see later, for K and t large enough, $D\xi(t)$ is bounded away from 0.

Let $\bar{\gamma} = \max_{\theta\in[-\pi,\pi]}\omega'(\theta)$ [clearly $\bar{\gamma} > 0$ and $\bar{\gamma} = -\min_{\theta\in[-\pi,\pi]}\omega'(\theta)$]. Choose $\gamma > \bar{\gamma}$ and consider the sets

$$I(t) = \{j \in \mathbb{Z} : [-\gamma t] \leqslant j \leqslant [\gamma t]\}$$
$$I_r(t) = \{j \in \mathbb{Z} : (2r-1)\beta(t) + \gamma\delta(t) \leqslant j \leqslant (2r+1)\beta(t) + \gamma\delta(t)\}$$
$$J(t) = I(t)\Big\backslash \bigcup_{r=-\kappa(t)}^{\kappa(t)} I_r(t)$$

where $\kappa(t) = \{[\gamma t - \beta(t)]/[2\beta(t) + \delta(t)]\}$ and corresponding sums are

$$A_r(t) = \sum_{m\in I_r(t)} a_m(t)$$

$$\xi_1(t) = \sum_{m\in\mathbb{Z}\backslash I(t)} a_m(t)$$

$$\xi_2(t) = \sum_{m\in J(t)} a_m(t)$$

We have $\xi(t) = \bar{\xi}(t) + \xi_1(t) + \xi_2(t)$ with $\bar{\xi}(t) = \sum_{r=-\kappa(t)}^{\kappa(t)} A_r(t)$ and

$$|\xi_1(t)| \leqslant \sum_{m\in\mathbb{Z}\backslash I(t)} |a_m(t)| \leqslant K \sum_{m\in\mathbb{Z}\backslash I(t)} R_m(t)$$

$$\mathbb{E}_P\left(\left[\xi_2(t)\right]^2\right) \leqslant \sum_{m\in J(t)} \mathbb{E}_P\left(\left[a_m(t)\right]^2\right) + \sum_{m,m'\in J(t)}{}' \left|\mathbb{E}_P\left(a_m(t)a_{m'}(t)\right)\right|$$
$$\leqslant t^{-2/3}|J(t)|K^2c^2\left[1 + 8\sum_{j>0}\alpha_P(j)\right] = c_7 t^{-2/3}|J(t)|$$

where $|J(t)|$ is the cardinality of the set $J(t)$ and we made use of Proposition 2.1 and inequality (3.6). By Proposition A.2

$$\lim_{t\to+\infty}\sum_{m\in\mathbb{Z}\setminus I(t)} R_m(t)=0$$

and, since $|J(t)|\leqslant 2[\beta(t)+\kappa(t)\delta(t)]$, by conditions (a) and (b) on the functions β and δ, $\lim_{t\to+\infty}t^{-2/3}|J(t)|=0$. Therefore the random variable $\xi_1(t)+\xi_2(t)$ tends to 0 in L_2-norm as $t\to+\infty$ and the central limit theorem for $\xi(t)$ holds if it holds for $\bar{\xi}(t)$.

Using Proposition 2.1 $2\kappa(t)$ times, and recalling condition (c) on the functions β and δ, we have, for all $\tau\in\mathbb{R}^1$,

$$\left|\mathbb{E}_P\left(\exp\left[i\tau\bar{\xi}(t)\right]\right)-\prod_{r=-\kappa(t)}^{\kappa(t)}\mathbb{E}_P\left(\exp\left[i\tau A_r(t)\right]\right)\right|$$
$$\leqslant 16\left[2\kappa(t)+1\right]\alpha_P(\delta(t))\to 0 \qquad \text{as} \quad t\to+\infty$$

Therefore if we introduce the r.v. $\tilde{\xi}(t)=\sum_{r=-\kappa(t)}^{\kappa(t)}\tilde{A}_r(t)$, where the new r.v.'s $\tilde{A}_r(t)$, $r=-\kappa(t),\ldots,\kappa(t)$, are independent and $\tilde{A}_r(t)$ has the same distribution of $A_r(t)$, its limiting distribution, if it exists, coincides with the limiting distribution of $\bar{\xi}(t)$. Moreover, using the infinitesimality of $\xi_1(t)+\xi_2(t)$ we have

$$\begin{aligned}
|D\xi(t)-D\tilde{\xi}(t)| &\leqslant |D\xi(t)-D\bar{\xi}(t)|+|D\bar{\xi}(t)-D\tilde{\xi}(t)|\\
&=o(1)+\left|\sum_{r,r'=-\kappa(t)}^{\kappa(t)}\mathbb{E}_P\left(A_r(t)A_{r'}(t)\right)\right|\\
&\leqslant o(1)+4K^2\sum_{\substack{j,m\in\mathbb{Z}\\ |j|>\delta(t)}} R_m(t)R_{m+j}(t)\alpha_P(|j|)\\
&=o(1)+4dK^2\sum_{|j|>\delta(t)}\alpha_P(|j|)=o(1)
\end{aligned}$$

We can now prove that the Lindeberg condition for the r.v. $\tilde{\xi}(t)$ holds, i.e., that

$$\lim_{t\to+\infty}\sum_{r=-\kappa(t)}^{\kappa(t)}\int_{|z|>\epsilon(D\tilde{\xi}(t))^{1/2}} z^2\,dF_r^t=0 \tag{3.7}$$

for any $\epsilon>0$, where F_r^t denotes the distribution function of the r.v. $A_r(t)$. This implies that the r.v. $\tilde{\xi}(t)/[D\tilde{\xi}(t)]^{1/2}$ tends in distribution to the normal law, and hence the same is true for the r.v. $\xi(t)/[D\xi(t)]^{1/2}$.

In order to prove that Eq. (3.7) holds we first prove that there is an absolute constant c_8 for which

$$\max_{|r|\leqslant\kappa(t)}\mathbb{E}_P\left(\left[A_r(t)\right]^6\right)\leqslant c_8\beta^3(t)t^{-2} \tag{3.8}$$

It is easy to see that

$$\mathbb{E}_P\big([A_r(t)]^6\big) = H_r(t) + \sum_{k_1, \ldots, k_4 \in I_r(t)} \mathbb{E}_P\big(a_{k_1}^3 a_{k_2} a_{k_3} a_{k_4} + a_{k_1}^2 a_{k_2}^2 a_{k_3} a_{k_4}\big)$$

$$+ \sum_{k_1, \ldots, k_5 \in I_r(t)}{}' \mathbb{E}_P\big(a_{k_1}^2 a_{k_2} \ldots a_{k_5}\big)$$

$$+ \sum_{k_1, \ldots, k_6 \in I_r(t)}{}' \mathbb{E}_P(a_{k_1}, \ldots, a_{k_6})$$

where $H_r(t)$ is a sum of three-, two-, and one-index terms, each of which is less than $K^6 c^6 [2\beta(t)]^3 t^{-2}$. We now show how to prove such an estimate for the six-index term. For the other terms the proof is similar. We can consider only the sum for $k_1 < k_2 < \cdots k_6$. Proceeding in analogy with the proof of inequality (3.1) we set $k_1 = k$, $k_{i+1} - k_i = j_i$, $i = 1, \ldots, 5$, and split the sum according to which of the j_i's is the largest. We get for instance the estimate

$$\sum_{\substack{k, j_1, \ldots, j_5 \\ \max(j_1, \ldots, j_4) \leqslant j_5}}^{(r)} \big|\mathbb{E}_P(a_k a_{k+j_1} \ldots a_{k+j_1+j_2+\cdots+j_5})\big|$$

$$\leqslant 4K^6 c^6 t^{-2} \sum_{\substack{k, j_1 \ldots j_5 \\ \max(j_1 \ldots j_4) \leqslant j_5}}^{(r)} \alpha_P(j_5)$$

$$\leqslant 4K^6 c^6 t^{-2} [2\beta(t)]^3 \sum_{j>0} j^2 \alpha_P(j) = c_8' t^{-2} [\beta(t)]^3$$

where $\sum_{k, j_1, \ldots, j_5}^{(r)}$ denotes the sum over all indices $j_1, \ldots, j_5$ such that $j_i > 0$ and $k, k + j_i \in I_r(t)$, $i = 1, \ldots, 5$. For the other orderings of the j_i's, terms containing products of expectations may appear, which are treated by repeated application of the same method.

Since $D\xi(t)$ is bounded away from 0 for t large enough, so is $D\tilde{\xi}(t)$, so that we have

$$\int_{|z| > \epsilon [D\tilde{\xi}(t)]^{1/2}} z^2 \, dF_r^t(z) \leqslant \frac{1}{\epsilon^4 [D\tilde{\xi}(t)]^2} \int z^6 \, dF_r^t(z) \leqslant \frac{c_9}{\epsilon^4} t^{-2} \beta^3(t)$$

and hence

$$\sum_{r=-\kappa(t)}^{\kappa(t)} \int_{|z| > \epsilon [D\tilde{\xi}(t)]^{1/2}} z^2 \, dF_r^t(z) \leqslant [2\kappa(t) + 1] \frac{c_9}{\epsilon^4} t^{-2} \beta^3(t)$$

$$= O\big(\beta^2(t) t^{-1}\big) = o(1)$$

which shows that the Lindeberg condition is satisfied.

Consider now the original variable $q_0(t)$. Since $\xi(t) = \sum_{k \in \mathbb{Z}} [\mathfrak{A}_k(t) y_k^{(K)}]^{(1)}$, setting $\bar{a}_k^{(K)}(t) = [\mathfrak{A}_k(t) \bar{x}_k^{(K)}]^{(1)}$ and $\eta^{(K)}(t) = \sum_{k \in \mathbb{Z}} [\bar{a}_k^{(K)}(t) -$

$\mathbb{E}_P(\bar{a}_k^{(K)}(t))]$, we can write $q_0(t) = \xi(t) + \eta^{(K)}(t)$. We have, applying again Proposition 2.2

$$\begin{aligned}|\mathbb{E}_P\big(\bar{a}_j^{(K)}(t)\bar{a}_h^{(K)}(t)\big) &- \mathbb{E}_P a_j^{(K)}(t)\mathbb{E}_P a_h^{(K)}(t)| \\ &\leqslant 10\alpha_P^{1/2}(|j-h|)R_j(t)R_h(t)\|\bar{x}_j^{(K)}\|_4\|\bar{x}_h^{(K)}\|_4 \\ &\leqslant 10(c_2^2/K^{\lambda/2})\alpha_P^{1/2}(|j-h|)R_j(t)R_h(t)\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{E}_P\big(\eta^{(K)}(t)\big)^2 = \sum_{k\in\mathbb{Z}}\mathbb{E}_P\big(\bar{a}_k^{(K)}(t) - \mathbb{E}_P\bar{a}_k^{(K)}(t)\big)^2 + \sum_{j,h\in\mathbb{Z}}{}'\big(\mathbb{E}_P\,\bar{a}_j^{(K)}(t)\bar{a}_h^{(K)}(t)\big) \\ - \mathbb{E}_P\big(\bar{a}_j^{(K)}(t)\mathbb{E}_P\bar{a}_h^{(K)}(t)\big) \leqslant c_{10}/K^{2+\lambda} + c_{11}/K^{\lambda/2}\end{aligned}$$

which proves that $\eta^{(K)}(t)$ is uniformly small in L_2 sense. It follows in particular that for K large enough the dispersion $D\xi(t) = \mathbb{E}_P(\xi(t))^2$ is as close as we want to $Dq_0(t) = \sigma^2(t)$, and, since $\sigma^2(t)$ tends to a positive limit, $D\xi(t)$ will be eventually positive. By the previous result we have

$$\lim_{t\to\infty}\mathbb{E}_P\exp\big\{i\tau\xi(t)/\big[D\xi(t)\big]^{1/2}\big\} = \exp(-\tau^2/2)$$

But, if t is large enough

$$\begin{aligned}|\mathbb{E}_P\Big(\exp\big\{i\tau\xi(t)/\big[D\xi(t)\big]^{1/2}\big\}\Big) &- \exp\big[i\tau q_0(t)/\sigma(t)\big]| \\ &\leqslant |\tau|\,\mathbb{E}_P|\xi(t)/\big[D\xi(t)\big]^{1/2} - q_0(t)/\sigma(t)| \\ &\leqslant\big[|\tau|/\sigma(t)\big]\|\eta^{(K)}(t)\|_2 + |\tau|\,|\big[D\xi(t)\big]^{1/2} - \sigma(t)|\,\|\xi(t)\|_2\end{aligned}$$

This implies that $\lim_{t\to\infty}\mathbb{E}_P\exp[i\tau q_0(t)/\sigma(t)] = \exp(-\tau^2/2)$, for all $\tau\in\mathbb{R}^1$, and proves that the r.v. $q_0(t)$ converges in distribution to a Gaussian with mean 0 and dispersion σ^2.

Now the same considerations can be used to prove that any finite linear combination $\sum_{i=1}^n s_i(\mathfrak{U}_t x)_{k_i}^{(\alpha_i)}$, $s_i\in\mathbb{R}^1$, $k_i\in\mathbb{Z}$, $\alpha_i = 1,2$, $i = 1,\ldots,n$, converges in distribution to the appropriate Gaussian if the covariance converges. This implies that the state P_t converges weakly to G as $t\to+\infty$. Theorem 3.1 is proved. ■

To conclude the section we make some comments on the assumptions. Strong mixing, as a criterion of weak dependence of far away regions, is natural enough. The stronger condition of uniform strong mixing, which perhaps simplifies the proof, seems too restrictive, since, for instance, for a translation invariant Gaussian state it implies that far away variables are uncorrelated.[13] The interplay between the integrability condition and the decay rate of the mixing coefficient [Assumptions (ii) and (iii)] is clarified by Proposition 2.2, which is an essential ingredient of the theorem. Finally note that the zero mean condition (i) cannot be simply removed: if for instance in the initial state $\mathbb{E}_P x_j = a, j \in \mathbb{Z}$, for some $a \in \mathbb{R}^2$, $a \neq 0$, $\mathbb{E}_{P_t} x_0$ is a periodic function. In such a case one should consider the centered variables.

4. SUFFICIENT CONDITIONS FOR CONVERGENCE OF THE COVARIANCE

Theorem 4.1. Let c be a covariance such that

(i) $\quad |C^{(\alpha,\beta)}(h,h+k)| \leqslant \gamma(|k|), \qquad \alpha,\beta = 1,2; \quad h,k \in \mathbb{Z}$

where $\gamma : \overline{\mathbb{Z}}_+ \to \overline{\mathbb{R}}_+$ and $\sum_{k \in \mathbb{Z}_+} \gamma(k) < \infty$, and

(ii) $\quad \lim_{h \to +\infty} C(h,h+k) = \mathscr{S}_k^R, \qquad \lim_{h \to -\infty} C(h,h+k) = \mathscr{S}_k^L, \qquad k \in \mathbb{Z}$

Let $\mathcal{V}$ be a force matrix satisfying Assumptions I–III of Section 2, and $\hat{\mathcal{U}}(\theta,t)$ be the function matrix associated to $\mathcal{V}$ by Eq. (2.5). Then the covariance C_t, given by

$$C_t(h,k) = \sum_{j,j' \in \mathbb{Z}} \mathcal{U}_{h-j}(t) C(j,j') \mathcal{U}_{k-j'}^T(t) \tag{4.1}$$

converges, as $t \to +\infty$, in the sense of Definition 2.4, to the translation invariant covariance with spectral density $\hat{f} = \hat{f}^{(1)} + \hat{f}^{(2)}$, where

$$\hat{f}^{(1)}(\theta) = \tfrac{1}{2}\{\hat{\mathscr{S}}_+(\theta) + \hat{\mathcal{C}}(\theta)\hat{\mathscr{S}}_+(\theta)\hat{\mathcal{C}}^T(\theta)\}$$

$$\hat{f}^{(2)}(\theta) = i\,\mathrm{sgn}(\omega'(\theta))\{\hat{\mathscr{S}}_-(\theta)\hat{\mathcal{C}}^T(\theta) - \hat{\mathcal{C}}(\theta)\hat{\mathscr{S}}_-(\theta)\}$$

with

$$\hat{\mathcal{C}}(\theta) = \begin{pmatrix} 0 & 1/\omega(\theta) \\ -\omega(\theta) & 0 \end{pmatrix}$$

and $\hat{\mathscr{S}}_\pm(\theta)$ are the functions with Fourier coefficients $\mathscr{S}_k^\pm = \frac{1}{2}(\mathscr{S}_k^R \pm \mathscr{S}_k^L)$, $k \in \mathbb{Z}$, respectively.

Before going to the proof, we remark that the limiting covariance is stationary under the dynamics induced by $\mathcal{V}$ (see Ref. 11). Namely, $\hat{f}^{(1,1)}(\theta)$ and $\hat{f}^{(2,2)}(\theta)$ are real even functions and $\hat{f}^{(1,1)}(\theta) = \omega^{-2}(\theta)\,\hat{f}^{(2,2)}(\theta)$, while $\hat{f}^{(1,2)}(\theta)$ is a purely imaginary odd function and $\hat{f}^{(1,2)}(\theta) = -\hat{f}^{(2,1)}(\theta)$. In fact, since $\mathscr{S}^\Pi$ ($\Pi = R,L$) is the covariance of a real process, the corresponding spectral density satisfied the relations

$$\hat{\mathscr{S}}_\Pi(\theta) = \overline{\hat{\mathscr{S}}_\Pi}(-\theta), \quad \hat{\mathscr{S}}_\Pi^{(1,2)}(\theta) = \hat{\mathscr{S}}_\Pi^{(2,1)}(-\theta), \quad \hat{\mathscr{S}}_\Pi^{(\alpha,\alpha)}(\theta) = \hat{\mathscr{S}}_\Pi^{(\alpha,\alpha)}(-\theta), \quad \alpha = 1,2$$

Proof. Setting $C^{(1)}(h,k)=\mathfrak{s}_{k-h}^{+}$ and

$$C^{(2)}(h,k)=\begin{cases}\mathfrak{s}_{k-h}^{-} & \text{for} \quad h>0\\ 0 & \text{for} \quad h=0\\ -\mathfrak{s}_{k-h}^{-} & \text{for} \quad h<0\end{cases}$$

$h,k\in\mathbb{Z}$, we can write $C=C^{(1)}+C^{(2)}+C^{(3)}$, and, correspondingly, $C_t=C_t^{(1)}+C_t^{(2)}+C_t^{(3)}$. We show first that $\lim_{t\to\infty}C_t^{(3)}=0$. We fix $h,k\in\mathbb{Z}$ and set $\mathfrak{F}_j(t)=\sum_{n\in\mathbb{Z}}\mathfrak{U}_{h-n}(t)C^{(3)}(n,n+j)\mathfrak{U}_{k-n-j}^{T}(t)$ so that

$$C_t^{(3)}(h,k)=\sum_{j\in\mathbb{Z}}\mathfrak{F}_j(t) \tag{4.2}$$

From assumptions (i) and (ii) it follows that $|(\mathfrak{s}_k^{\pm})^{(\alpha,\beta)}|\leqslant\gamma(|k|)$ and $|(C^{(3)}(h,h+k))^{(\alpha,\beta)}|\leqslant 2\gamma(|k|)$, $k\in\mathbb{Z}_+$. Setting, as in Section 2, $d=\sup_{t\in\mathbb{R}^1}\sum_{k\in\mathbb{Z}}R_k^2(t)$ [see (2.9)], we get for all $t\in\mathbb{R}^1$

$$|(\mathfrak{F}_j(t))^{(\alpha,\beta)}|\leqslant 2\gamma(|j|)\sum_{l\in\mathbb{Z}}R_{h-l}(t)R_{k-l-j}(t)\leqslant 2d\gamma(|j|), \qquad \alpha,\ \beta=1,2$$

which shows that the series (4.2) converges uniformly in t. Therefore it is enough to prove that $\lim_{t\to+\infty}\mathfrak{F}_j(t)=0$ for each $j\in\mathbb{Z}$. Since $\lim_{n\to\pm\infty}C(n,n+j)=0$, it is possible to find $N\in\mathbb{Z}_+$ so large that $|(C^{(3)}(n,n+j))^{(\alpha,\beta)}|\leqslant\epsilon/d$ for $n\geqslant N$, which implies $\sum_{|l|<N}|(\mathfrak{U}_{h-l}(t)C^{(3)}(l,l+j)\mathfrak{U}_{k-l-j}^{T}(t))^{(\alpha,\beta)}|\leqslant\epsilon$. On the other hand inequality (3.6) implies that $\lim_{t\to\infty}\sum_{|l|<N}\mathfrak{U}_{h-l}(t)C^{(3)}(l,l+j)\mathfrak{U}_{k-l-j}^{T}(t)=0$, which proves that $\lim_{t\to\infty}\mathfrak{F}_j(t)=0$.

The next step consists in proving that

$$\lim_{t\to\infty}C_t^{(1)}(h,h+k)=f_k^{(1)}, \qquad k\in\mathbb{Z} \tag{4.3}$$

We have

$$\begin{aligned}C_t^{(1)}(h,h+k)&=\sum_{l,l'\in\mathbb{Z}}\mathfrak{U}_{-l}(t)\mathfrak{s}_{l-l'}^{+}\mathfrak{U}_{k-l'}^{T}(t)\\&=\sum_{l,l'\in\mathbb{Z}}\mathfrak{U}_{l}(t)\mathfrak{s}_{l'-l}^{+}\mathfrak{U}_{k-l'}^{T}(t)\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ik\theta}\hat{\mathfrak{U}}(\theta,t)\hat{\mathfrak{s}}_{+}(\theta)\hat{\mathfrak{U}}^{T}(\theta,t)\,d\theta\end{aligned}$$

[$\hat{\mathfrak{U}}(\theta,t)$ is even in θ]. This shows by the way that $C_t^{(1)}$ is translation invariant. Denoting by $\mathfrak{I}$ the identity matrix we have

$$\hat{\mathfrak{U}}(\theta,t)=\cos[\omega(\theta)t]\mathfrak{I}+\sin[\omega(\theta)t]\hat{\mathfrak{C}}(\theta)$$

and therefore

$$\begin{aligned}\hat{\mathfrak{U}}(\theta,t)\hat{\mathfrak{s}}_{+}(\theta)\hat{\mathfrak{U}}^{T}(\theta,t)&=\cos^2[\omega(\theta)t]\hat{\mathfrak{s}}_{+}(\theta)+\sin^2[\omega(\theta)t]\hat{\mathfrak{C}}(\theta)\hat{\mathfrak{s}}_{+}(\theta)\hat{\mathfrak{C}}^{T}(\theta)\\&\quad+\tfrac{1}{2}\sin[2\omega(\theta)t]\{\hat{\mathfrak{s}}_{+}(\theta)\hat{\mathfrak{C}}^{T}(\theta)+\hat{\mathfrak{C}}(\theta)\hat{\mathfrak{s}}_{+}(\theta)\}\end{aligned}$$

Using simple trigonometrics and observing that (Proposition A.1) terms containing the factor $\exp[i2\omega(\theta)t]$ vanish in the limit $t \to \infty$, we get

$$\lim_{t\to\infty} C_t^{(1)}(h, h+k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{-ik\theta}\tfrac{1}{2}\{\hat{\mathfrak{s}}_+(\theta) + \hat{\mathcal{C}}(\theta)\hat{\mathfrak{s}}_+(\theta)\hat{\mathcal{C}}^T(\theta)\}$$

which proves Eq. (4.3). Note that this concludes the proof if the initial state is translation invariant in the wide sense.

The final step consists in proving that

$$\lim_{t\to+\infty} C_t^{(2)}(h, h+k) = f_k^{(2)}, \qquad k \in \mathbb{Z} \tag{4.4}$$

Let $h, k \in \mathbb{Z}$ be fixed. We have

$$C_t^{(2)}(h,k) = \sum_{l>0}\sum_{l'\in\mathbb{Z}} \mathfrak{U}_{h-l}(t)\mathfrak{s}_{l'-l}^{-}\mathfrak{U}_{k-l'}^T(t) - \sum_{l<0}\sum_{l'\in\mathbb{Z}} \mathfrak{U}_{h-l}(t)\mathfrak{s}_{l'-l}^{-}\mathfrak{U}_{k-l'}^T$$

Since, as is easily seen,

$$\sum_{l'\in\mathbb{Z}} \mathfrak{U}_{h-l}(t)\mathfrak{s}_{l'-l}^{-}\mathfrak{U}_{h+k-l'}^T(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{il\theta}\hat{\mathcal{G}}(\theta,t) \equiv \mathcal{G}_{-l}(t)$$

with

$$\hat{\mathcal{G}}(\theta,t) = \frac{e^{-ih\theta}}{2\pi}\int_{-\pi}^{\pi} d\varphi\, e^{-ik\varphi}\hat{\mathfrak{U}}(\theta-\varphi,t)\hat{\mathfrak{s}}_-(\varphi)\hat{\mathfrak{U}}^T(\varphi,t)$$

we find $C_t^{(2)}(h,h+k) = \sum_{l<0}\mathcal{G}_l(t) - \sum_{l>0}\mathcal{G}_l(t) = 2i\tilde{\mathcal{G}}(0,t)$, where $\tilde{\mathcal{G}}$ is the harmonic conjugate of $\hat{\mathcal{G}}$. We have (see Ref. 14)

$$\tilde{\mathcal{G}}(0,t) = \frac{1}{\pi}\int_0^{\pi} \frac{\hat{\mathcal{G}}(\theta,t) - \hat{\mathcal{G}}(-\theta,t)}{2\tan(\theta/2)}\, d\theta = \frac{1}{2\pi} P\int_{-\pi}^{\pi} \hat{\mathcal{G}}(\theta,t)\cot\!\mathrm{an}(\theta/2)\, d\theta$$

$$= \frac{1}{2\pi^2}\int_{-\pi}^{\pi} d\varphi \left[P\int_{-\pi}^{\pi} \frac{\hat{\mathfrak{U}}(\theta-\varphi,t)}{2\tan(\theta/2)}\, e^{-ih\theta}\right]\hat{\mathfrak{s}}_-(\theta)\hat{\mathfrak{U}}^T(\varphi,t)e^{-ik\varphi}$$

where P denotes the Cauchy principal part. By Proposition A.4 the integrals

$$P\int_{-\pi}^{\pi} d\theta\, e^{-ih\theta}\frac{(\hat{\mathfrak{U}}(\theta-\varphi,t))^{(\alpha,\beta)}}{2\tan(\theta/2)}, \qquad \alpha,\beta = 1,2$$

are uniformly bounded in φ, and have the following expansion for large t:

$$P\int_{-\pi}^{\pi} d\theta \frac{\hat{\mathfrak{U}}(\theta-\varphi,t)}{2\tan(\theta/2)}\, e^{-ih\theta} \sim \pi\, \mathrm{sgn}[\omega'(-\varphi)]$$

$$\times\{-\sin[\omega(\varphi)t]\mathfrak{T} + \cos[\omega(\varphi)t]\hat{\mathcal{C}}(\varphi)\}$$

Therefore

$$\lim_{t\to+\infty} \tilde{\mathcal{G}}(0,t) = \lim_{t\to\infty} \frac{i}{\pi} \int_{-\pi}^{\pi} d\varphi \, \mathrm{sgn}\left[\omega'(\varphi)\right] e^{-ih\varphi}$$
$$\times \left\{\sin\left[\omega(\varphi)t\right]\mathcal{T} - \cos\left[\omega(\varphi)t\right]\mathcal{C}(\varphi)\right\}$$
$$\times \hat{s}_-(\varphi)\left\{\cos\left[\omega(\varphi)t\right]\mathcal{T} + \sin\left[\omega(\varphi)t\right]\hat{\mathcal{C}}^T(\varphi)\right\}$$
$$= \frac{i}{2\pi}\int_{-\pi}^{\pi} d\varphi \, e^{-ik\varphi}\left[\hat{s}_-(\varphi)\hat{\mathcal{C}}^T(\varphi) - \hat{\mathcal{C}}(\varphi)\hat{s}_-(\varphi)\right]\mathrm{sgn}\left[\omega'(\varphi)\right]$$

which proves Eq. (4.4). Theorem 4.1 is proved. ■

As an example, we deduce the limiting covariance the (generalized) Lebowitz–Spohn case, i.e., in the case in which the initial state far away to the left and to the right is an equilbrium state with temperature T_L, T_R respectively. In such a case the spectral functions corresponding to the left and right covariances are

$$\hat{s}_{\Pi}(\theta) = T_{\Pi}\begin{pmatrix} 1/\omega^2(\theta) & 0 \\ 0 & 1 \end{pmatrix}$$

($\pi = L, R$), which gives

$$\hat{s}_+(\theta) = \overline{T}\begin{pmatrix} 1/\omega^2(\theta) & 0 \\ 0 & 1 \end{pmatrix}, \qquad \hat{s}_-(\theta) = \Delta T\begin{pmatrix} 1/\omega^2(\theta) & 0 \\ 0 & 1 \end{pmatrix}$$

where $T = (T_R + T_L)$, $\Delta T = (T_R - T_L)$. Therefore we have

$$\hat{f}^{(1)}(\theta) = \hat{s}_+(\theta) = \overline{T}\begin{pmatrix} 1/\omega^2(\theta) & 0 \\ 0 & 1 \end{pmatrix},$$
$$\hat{f}^{(2)}(\theta) = \Delta T i \, \mathrm{sgn}\left[\omega'(\theta)\right]\begin{pmatrix} 0 & -1/\omega(\theta) \\ 1/\omega(\theta) & 0 \end{pmatrix}$$

and we see that the diagonal terms are proportional to the average temperature, whereas the off-diagonal ones (related to the heat flux) are proportional to the temperature jump.

Note, finally, that the limit for $t \to -\infty$ is different, and is obtained by changing the sign of $\hat{f}^{(2)}$.

The following result gives sufficient conditions for convergence in the periodic case.

Theorem 4.2. Let the covariance C satisfy assumption (i) of Theorem 4.1, and suppose moreover that there is a positive integer N such that $C(h,k) = C(h+N, k+N)$. Then, if the function $\omega(\theta)$ is not $(2\pi/n)$-periodic for any divisor n of N, the covariance C_t, $t \in \mathbb{R}^1$, given by Eq.

(4.1), converges, as $t \to +\infty$, to the translation invariant covariance with spectral density

$$\hat{f}(\theta) = \tfrac{1}{2}\left[h(\theta) + \hat{\mathcal{C}}(\theta)h(\theta)\hat{\mathcal{C}}^T(\theta)\right]$$

where

$$h(\theta) = \frac{1}{N}\sum_{r,r'=0}^{N-1} e^{i\theta(r-r')}g_{r,r'}^{(\theta)} \quad \text{and} \quad g_{r,r'}(\theta) = \sum_{j\in\mathbb{Z}} C(r, jN + r')e^{ijN\theta}$$

Proof. Note that the matrix $\{g_{r,r'}\}_{r,r'=0}^{N-1}$ is nonnegative definite, i.e.,

$$\sum_{r,r'=1}^{N-1}(a_r, g_{r,r'}, a_{r'}) = \sum_{r,r'=0}^{N-1}\sum_{\substack{\alpha=1,2\\ \beta=1,2}} \bar{a}_r^{(\alpha)}g_{r,r'}^{(\alpha,\beta)}a_{r'}^{(\beta)} \geqslant 0$$

Therefore $h(\theta)$ is a nonnegative definite function matrix, and so is $\hat{f}(\theta)$. Since

$$C_t(h, h+k) = \sum_{l,l'\in\mathbb{Z}}\sum_{r,r'=0}^{N-1} \mathfrak{A}_{h-lN-r}(t)C(r,(l'-l)N + r')\mathfrak{A}_{h+k-l'N-r'}^T(t)$$

setting $\hat{\mathfrak{A}}^{(p)}(\theta,t) = e^{-ip\theta}\hat{\mathfrak{A}}(\theta,t)$ we find

$$C_t(h, h+k) = \sum_{r,r'=0}\sum_{l,j\in\mathbb{Z}} \mathfrak{A}_{lN}^{(r-k)}(t)(g_{r,r'})_j\left(\mathfrak{A}_{-lN-j}^{(h+k-r')}(t)\right)^T$$

Now, let $\hat{g}: \mathbb{R}^1 \to \mathbb{C}$ be a function of period 2π and set

$$_{(N)}\hat{g}(\theta) = \frac{1}{N}\sum_{s=0}^{N-1}\hat{g}\left(\frac{\theta}{N} + \frac{2\pi s}{N}\right)$$

It is easy to see that the Fourier coefficients of $\hat{g}$ and $_{(N)}\hat{g}$ are related by $_{(N)}g_k = g_{kN}$, $k \in \mathbb{Z}$. Therefore we find, observing that $g_{r,r'}$ is $(2\pi/N)$-periodic,

$$C_t(h, h+k) = \sum_{r,r'=0}^{N-1}\frac{1}{2\pi}\int_0^{2\pi} {}_{(N)}\mathfrak{A}^{(r-h)}(\theta,t)g_{r,r'}\left(\frac{\theta}{N}\right)\left({}_{(N)}\mathfrak{A}^{(h+k-r')}(\theta,t)\right)^T d\theta \tag{4.5}$$

In the integral on the right-hand side all terms have phase factors of the type $\exp\{it[\omega(\theta/N + 2\pi s/N) \pm \omega(\theta/N + 2\pi s'/N)]\}$. If s and s' are such that the phase factor is not identically unity, the corresponding term vanishes as $t \to \infty$. Therefore

$$\lim_{t \to +\infty} C_t(h, h+k) = \frac{1}{N} \sum_{s=0}^{N-1} \int_0^{2\pi} \hat{f}(\theta/N + 2\pi s/N) \times \exp\left[-ik(\theta/N + 2\pi s/N)\right] d\theta$$

$$= \int_0^{2\pi} \hat{f}(\theta) \exp(-ik\theta)\, d\theta$$

which proves the theorem. ■

If Ω is $(2\pi/n)$-periodic, with n a divisor of N, the covariance converges in general to a limiting covariance which is not translation invariant. This can be easily understood since only oscillators at a distance which is a multiple of n interact. More precisely, the following result holds.

Corollary 4.3. Suppose that C satisfies the assumptions of Theorem 4.2 and that ω is a $(2\pi/n)$-periodic function, where n is a divisor of N. Then the covariance C_t, $t \in \mathbb{R}^1$, given by Eq. (4.5), converges to a limiting covariance C_∞ which is such that $C_\infty(h,k) = C_\infty(h+n, k+n)$, $h, k \in \mathbb{Z}$.

Proof. Carrying out the limit in Eq. (4.5) we have, setting $m = N/n$,

$$\lim_{t \to \infty} C_t(h, h+k)$$

$$= \frac{1}{2\pi N^2} \sum_{r,r'=0}^{N-1} \sum_{\substack{s,s'=0 \\ s-s'=jm, j \in \mathbb{Z}}}^{N-1} \int_0^{2\pi} \exp(-ik\theta/N)$$

$$\times \exp\left[-i(r+k-r')\theta/N\right]$$

$$\times \exp\left\{-i2\pi\left[s + h(s'-s) + s'(k-r')\right]/N\right\}$$

$$\times \left\{ g_{r,r'}(\theta/N) + \hat{\mathcal{C}}(\theta/N + 2\pi s/N) \times g_{r,r'}(\theta/N) \hat{\mathcal{C}}^T(\theta/N + 2\pi s'/N) \right\} d\theta$$

Clearly the right-hand side is n-periodic. ■

As a consequence, using the main theorem of Section 3, it is easy to provide examples of states which converge, as $t \to \infty$, under the action of a dynamics satisfying the assumptions of Corollary 4.3, to a non-translation-invariant limiting state. In view of the remark above such examples are trivial.

5. CONSTANTS OF THE MOTION AND STATIONARY STATES

It is natural to expect that infinite harmonic systems somehow inherit constants of the motion of the corresponding finite systems. We shall now make this statement precise.

Consider $2N+1$ oscillators of unitary mass with cyclic boundary conditions (i.e., on a circle), moving under the action of a translation invariant harmonic force. If (q_h, p_h), $h = -N, \ldots, N$, are the Hamiltonian variables and the force matrix is denoted by $\mathcal{V}: \mathcal{V}_{h,k} = V_{k-h} = V_{h-k}$, $h, k = -N, \ldots, N$, the system is conveniently described by the normal modes

$$Q_k = \frac{1}{(2N+1)^{1/2}} \sum_{h=-N}^{N} \exp(-i\theta_k h) q_h, \qquad k = -N, \ldots, N$$

where $\theta_k = 2\pi k/(2N+1)$. There is degeneration, since $\omega^2(\theta_k) = \sum V_h \exp(i\theta_k h) = \omega^2(\theta_{-k}) = \omega^2(-\theta_k)$, i.e., the normal modes Q_k and $Q_{-k} = \overline{Q}_k$ have the same frequency. Consider the constants of the motion $e(\theta_k) = \frac{1}{2}[|\dot{Q}_k|^2 + \omega^2(\theta_k)|Q_k|^2]$ (energy of the normal modes) and $a(\theta_k) = \frac{1}{2}(Q_k\dot{\overline{Q}}_k - \dot{Q}_k\overline{Q}_k)$ (angular momentum in the complex Q_k-plane: it is constant because the corresponding frequencies are degenerate), and their linear combinations

$$\begin{aligned} \tilde{e}_j^{(N)} &= \frac{1}{2\pi} \sum_{k=-N}^{N} e(\theta_k) \exp(-i\theta_k j) \Delta\theta_k, \\ a_j^{(N)} &= \frac{1}{2\pi} \sum_{k=-N}^{N} a(\theta_k) \exp(-i\theta_k j) \Delta\theta_k \end{aligned} \tag{5.1}$$

$j = -N, \ldots, N$, where $\Delta\theta_k = 2\pi/(2N+1)$. The physical meaning of the new constants of the motion appears from Eq. (5.1). They are the Fourier components of the energy density and of the "angular momentum density" of the normal modes in $[-\pi, \pi]$ (which is the Brillouin zone of our system). To go over to infinite systems it is convenient to write them in a different way. The finite system under consideration can be equivalently described by an infinite periodic chain of period $2N+1: x = \{(q_k, p_k)\}_{k \in \mathbb{Z}}$ with $x_{h+(2N+1)n} = x_h$, $h = -N, \ldots, N$, $n \in \mathbb{Z}$. Denoting again by $\mathcal{V}$ the new force matrix (which is translation invariant and of range N), and by $q = \{q_k\}_{k \in \mathbb{Z}}$ the sequence of the oscillator positions, it is easily seen that

$$\tilde{e}_j^{(N)} = \frac{1}{2N+1} \sum_{k=-N}^{N} \frac{1}{2} \left[p_h p_{h+j} + q_h (\mathcal{V}q)_{h+j} \right] \tag{5.2}$$

$$a_j^{(N)} = \frac{1}{2N+1} \sum_{k=-N}^{N} \frac{1}{2} (q_h p_{h+j} - p_h q_{h+j}) \tag{5.3}$$

In the general case, if $x \in \mathcal{X}'$ is not $(2N+1)$-periodic (and $\mathcal{V}$ has infinite range), these quantities are not conserved. However, as we shall presently show, we can get actual constants of the motion by studying their limits as $N \to \infty$. Note that $e_0^{(N)}$ is the average energy per oscillator in the box $[-N, N]$, and its limit as $N \to \infty$ is the usual specific total energy.

Actually, for technical reasons, instead of considering the quantities $\tilde{e}_j^{(N)}$ given by Eq. (5.2), we shall examine the quantities

$$e_j^{(N)} = \frac{1}{2N+1} \sum_{k=-N}^{N} \frac{1}{2} \left[p_h p_{h+j} + (\mathcal{V}^{1/2} q)_h (\mathcal{V}^{1/2} q)_{h+j} \right] \qquad (5.2')$$

Note that if the limits $\lim_{N\to\infty} e_j^{(N)}$, $j \in \mathbb{Z}$, exist, then the sequences $\{\tilde{e}_j^{(N)}\}_{N\in\mathbb{Z}_+}$, $j \in \mathbb{Z}$ converge to the same limit (see remark at the end of Theorem 5.1).

We shall first show that the quantities e_k, a_k, $k \in \mathbb{Z}$, formally defined as

$$e_k(x) = \lim_{N\to\infty} e_k^{(N)}(x) \qquad (5.4)$$

$$a_k(x) = \lim_{N\to\infty} a_k^{(N)}(x) \qquad (5.4')$$

where $e_k^{(N)}$, $a_k^{(N)}$, $k \in \mathbb{Z}$, are given by Eqs. (5.2′) and (5.3), are constants of the motion for the infinite harmonic chain.

We consider first the case of square summable initial data. Set

$$S = \begin{pmatrix} \mathcal{V} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \qquad A = GS$$

If $\mathcal{V}$ satisfies Assumptions I–III of Section 2, A, G, and S are bounded operators on $l_2 \oplus l_2$. Therefore, if T denotes the shift on $\mathcal{X} : (Tx)_k = x_{k-1}$, $k \in \mathbb{Z}$, the functionals $\mathcal{E}_k(x)$, $\mathcal{A}_k(x)$, defined on $l_2 \oplus l_2$ by

$$\begin{aligned} \mathcal{E}_k(x) &= \tfrac{1}{2}(S^{1/2}x, T^{-k}S^{1/2}x) \\ &= \tfrac{1}{2}(x, T^{-k}Sx) = \tfrac{1}{2} \sum_{h\in\mathbb{Z}} \left(P_h P_{h+k} + q_h \sum_{l\in\mathbb{Z}} V_{h+k-l} q_l \right) \\ \mathcal{A}_k(x) &= \tfrac{1}{2}(x, T^{-k}Gx) = \tfrac{1}{2} \sum_{h\in\mathbb{Z}} (q_h p_{h+k} - p_h q_{h+k}) \end{aligned}$$

make sense. Note furthermore that the restriction of the evolution $\{\mathcal{U}_t, t \in \mathbb{R}\}$, defined on $\mathcal{X}'$ by Theorem 2.1, to $l_2 \oplus l_2$ is a strongly continuous one-parameter group of operators with generator A.

Lemma 5.1. Suppose that the force matrix $\mathcal{V}$ satisfies Assumptions I–III of Section 2. Then for all $t \in \mathbb{R}^1$ and all $x \in l_2 \oplus l_2$

$$\mathcal{E}_k(\mathcal{U}_t x) = \mathcal{E}_k(x), \qquad \mathcal{A}_k(\mathcal{U}_t x) = \mathcal{A}_k(x), \qquad k \in \mathbb{Z}$$

Proof. The assertion comes immediately from the relations

$$AT^kS + T^kSA = AT^kG + T^kGA = 0$$

which can be proved by inspection.

In order to prove the more general result we need the following observations:

(1) existence of $e_0(x)$ and strict positivity of $\mathcal{V}$ imply

$$\text{(i)} \qquad x_h^2 = o(|h|), \qquad \text{(ii)} \qquad \|P_N x\|^2 = O(N)$$

(2) if $\mathfrak{B}$ is a symmetric translation invariant operator on $l_2 \oplus l_2$ such that

$$|\mathfrak{B}_{m,m+k}| = |B_k| \leqslant c_1 e^{-c_2|k|}, \qquad m,k \in \mathbb{Z}$$

where c_1, c_2 are positive constants, then $\mathfrak{B}$ is extendible to $\mathcal{X}'$ and

$$\|[\mathfrak{B}, P_N]x\|^2 = o(N)$$

for every $x \in \mathcal{X}'$ such that $x_h^2 = o(|h|)$. [Here P_N, $N \in \mathbb{Z}_+$, denotes as usual the projector: $(P_N x)_h = x_h$ if $|h| \leqslant N$, $= 0$ otherwise.] The proof of assertion (1) is trivial, while assertion (2) is proved as follows. We have

$$([\mathfrak{B}, P_N]x)_k = \begin{cases} \sum\limits_{|h| \geqslant N} B_{k-h} x_h & \text{if} \quad |k| \leqslant N \\ \sum\limits_{|h| \leqslant N} B_{k-h} x_h & \text{if} \quad |k| > N \end{cases}$$

Now

$$\begin{aligned} \|[\mathfrak{B}, P_N]x\|^2 &= \sum_k ([\mathfrak{B}, P_n]x)_k^2 \\ &= \sum_{|k| \leqslant N} \Big(\sum_{|h| > N} B_{k-h} x_h \Big)^2 + \sum_{|k| > N} \Big(\sum_{|h| \leqslant N} B_{k-h} x_h \Big)^2 \\ &\leqslant \max_{|k| \leqslant N} \Big| \sum_h B_{k-h} x_h \Big| \sum_{\substack{|k| \leqslant N \\ |h| > N}} |B_{k-h}| \, |x_h| \\ &\quad + \max_{|h| \leqslant N} \sum_l |B_l| \sum_{\substack{|k| > N \\ |h| \leqslant N}} |B_{k-h}| \, |x_h| \end{aligned}$$

Observing that $(\mathfrak{B} x)_k^2 = o(|k|)$, and that $\sum_{s=0}^{\infty} |B_{N+S}|$ and $\sum_{S=-\infty}^{0} |B_{-N+S}|$ are exponentially decreasing in N we have the needed result.

Now we can prove the following theorem.

Theorem 5.1. Suppose that the force matrix $\mathcal{V}$ satisfies Assumptions I–III of Section 2, and let $\{\mathcal{U}_t,\ t \in \mathbb{R}^1\}$ be the evolution on $\mathcal{X}'$ associated to $\mathcal{V}$. Then, if $x \in \mathcal{X}'$is such that the limits $e_k(x)$ and $a_k(x)$, $k \in \mathbb{Z}$ exist, the limits $e_k(\mathcal{U}_t x)$, $a_k(\mathcal{U}_t x)$ also exist, and $e_k(\mathcal{U}_t x) = e_k(x)$, $a_k(\mathcal{U}_t x) = a_k(x)$, k for all $t \in \mathbb{R}^1$.

Proof. Setting $\mathcal{U}_t^{(N)} = [P_N, S^{1/2}\mathcal{U}_t]$ we find

$$\begin{aligned}
&\left(P_N S^{1/2}\mathcal{U}_t x, T^{-k} P_N S^{1/2}\mathcal{U}_t x\right) \\
&\quad = \left(S^{1/2}\mathcal{U}_t P_N x, T^{-k} S^{1/2}\mathcal{U}_t P_N x\right) + \left(S^{1/2}\mathcal{U}_t P_N x, T^{-k}\mathcal{U}_t^{(N)} x\right) \\
&\qquad + \left(\mathcal{U}_t^{(N)} x, T^{-k} S^{1/2}\mathcal{U}_t P_N x\right) + \left(\mathcal{U}_t^{(N)} x, T^{-k}\mathcal{U}_t^{(N)} x\right)
\end{aligned}$$

It follows from Lemma 5.1 that

$$\left(S^{1/2}\mathcal{U}_t P_N x, T^{-k} S^{1/2}\mathcal{U}_t P_N x\right) = \left(S^{1/2} P_N x, T^{-k} S^{1/2} P_N x\right)$$

Therefore in order to prove that e_k is constant it is enough to prove that the remaining terms are $o(N)$. This is easily done remembering that the evolution coefficients decrease exponentially (in the index), and making use of the estimates (1) and (2). For a_k the proof is similar. ■

Remark. From (1) and (2) it is easy to see that $(\tilde{e}_k^N - e_k^N) = o(1)$ as $N \to \infty$.

The following result shows that the subset of the points $x \in \mathcal{X}$ for which the limits $e_k(x), a_k(x)$, $k \in \mathbb{Z}$ exist is "large enough," i.e., it is a set of full measure with respect to a large class of states. The conditions on the states are convergence of the expected values $\mathbb{E}_P(e_k^N)$, $\mathbb{E}_P(a_k^N)$ of e_k^N, a_k^N, $k \in \mathbb{Z}$ as $N \to \infty$, plus a condition of uniform integrability and weak dependence as in Theorem 3.1.

Theorem 5.2. Suppose that $\mathcal{V}$ satisfies Assumptions I and II of Section 2, and let P be a state satisfying the assumptions of Theorem 3.1 and moreover such that the limits $\langle e_k \rangle = \lim_{N\to\infty} \mathbb{E}_P e_k^N$, $\langle a_k \rangle = \lim_{N\to\infty} \mathbb{E}_P a_k^N$ exist. Then for P-almost all points $x \in \mathcal{X}$ the limits $e_k(x)$, $a_k(x)$ exist and $e_k(x) = \langle e_k \rangle$, $a_k(x) = \langle a_k \rangle$.

Proof. We shall prove that the centered variables $\bar{e}_k^N = e_k^N - \mathbb{E}_P e_k^N$, $\bar{a}_k^N = a_k^N - \mathbb{E}_P a_k^N$ tend to 0 P-almost everywhere. The proof is based on an argument due to Serfling (Ref. 15, Sec. 3.7): it shows that almost sure convergence of arithmetic means of random variables follows from some mild restrictions on the second moment of their sums.

Let $k \in \mathbb{Z}$ be fixed and set

$$b_h = \left((S^{1/2}x)_h, (S^{1/2}x)_{h+k}\right), \qquad \bar{b}_h = b_h - \mathbb{E}_P b_h,$$

$$u_1^2 = \sum_{\alpha,\beta=1}^{2} \left[(T^{-k}S^{1/2})_l^{(\alpha,\beta)}\right]^1, \qquad l \in \mathbb{Z}$$

$(T^{-k}S^{1/2})_l$ denotes by abuse of notation the 2×2 matrix $(T^{-k}S^{1/2})_{0,1}$ (T commutes with S). We have $\bar{e}_k^N = [1/(2N+1)]\sum_{|h|\leqslant N} b_h$. The main point in the proof consists in showing that

$$|\mathbb{E}_P(\bar{b}_h \bar{b}_{h'})| \leqslant g(|h-h'|) \tag{5.5}$$

for some function $g : \overline{\mathbb{Z}}_+ \to \overline{\mathbb{R}}_+$ such that $\sum_{r=0}^{\infty} g(r) < \infty$. Clearly $\mathbb{E}_P(\bar{b}_h^2) \leqslant c_1 \sum_{l \in \mathbb{Z}} u_l$. Therefore, suppose for instance $h' - h = r > 0$ and set

$$\mathcal{F} = \{ l = (l_1, l_2, l_3, l_4) \in \mathbb{Z}^4 : \max(l_1, l_2) < [r/2] < \min(l_3, l_4) + r \}$$

we have

$$\begin{aligned} &\mathbb{E}_P(\bar{b}_h \bar{b}_{h+r}) \\ &\quad \leqslant \sum_{l \in \mathcal{F}} u_{h-l_1} u_{h+k-l_2} u_{h+r-l_3} u_{h+k+r-l_4} [\mathbb{E}_P(q_{l_1} q_{l_2} q_{l_3} q_{l_4}) - \mathbb{E}_P(q_{l_1} q_{l_2}) \mathbb{E}_P(q_{l_3} q_{l_4})] \\ &\quad + \sum_{l \in \mathcal{F}^c} u_{h-l_1} u_{h+k-l_2} u_{h+r-l_3} u_{h+k+r-l_4} [\mathbb{E}_P(q_{l_1} q_{l_2} q_{l_3} q_{l_4}) \\ &\qquad\qquad - \mathbb{E}_P(q_{l_1} q_{l_2}) \mathbb{E}_P(q_{l_3} q_{1_4})] \end{aligned}$$

Using Proposition 2.2 and with simple manipulations it is easily seen that the first term is less than

$$c_2 \sum_{l_1 \leqslant [r/2] \leqslant l_3 + r} u_{h-l_1} u_{h+r-l_3} \alpha_P^{\eta}(\beta(l_1, l_3)) \tag{5.6}$$

where $\beta(l_1, l_3) = \min(r, l_3 + r) - \max(0, l_1)$, $\eta = \delta/(4 + \delta)$. Now we have

$$(5.6) \leqslant c_3 (A^2 \alpha_P^{\mu}(r) + A(u * \gamma)_r + A(\tilde{u} * \gamma)_r + (u * \gamma * \tilde{u})_r) = g_1(r)$$

where $A = \sum_{k \in \mathbb{Z}} u_k$, $\gamma_k = \alpha_P^{\eta}(|k|)$ for $k \neq 0$ and $\gamma_0 = 0$, $\tilde{u}_k = u_{-k}$, $k \in \mathbb{Z}$, and $*$ denotes the convolution. Since $\{\gamma_k\}_{k \in \mathbb{Z}} \in l_1(\mathbb{Z})$ and $\{u_k\}_{k \in \mathbb{Z}}$ is of rapid decrease, g_1 is summable. The other term is estimated by

$$c_4 \sum_{l \in \mathcal{F}^c} u_{h-l_1} u_{h+k-l_2} u_{h+r-l_3} u_{h+k+r-l_4} \leqslant 4 c_4 A^3 \sum_{|l'| \geqslant [r/2]} u_{l'} = g_2(r)$$

Clearly $g_2(r)$ is summable and inequality (5.5) is proved. Now this inequality implies that for some absolute constant c_5

$$\mathbb{E}_P \left(\sum_{h=S}^{S+N} \bar{b}_h \right)^2 \leqslant c_5 N$$

This (linear) bound makes Serfling's theorem applicable (Ref. 15, Theorem 3.7.3); we conclude that $\lim_{N \to \infty} \bar{e}_k^N = 0$ P-a.e.. For the constants a_k, $k \in \mathbb{Z}$ the proof goes along the same lines.

An inspection of Theorems 4.1, 4.2 shows that the class of the limiting states which we obtain is the class of the translational invariant Gaussian states with zero means and spectral density given by

$$\hat{f}(\theta) = \frac{1}{2} \begin{bmatrix} \dfrac{\hat{g}(\theta)}{\omega^2(\theta)} & \hat{h}(\theta) \\ -\hat{h}(\theta) & \hat{g}(\theta) \end{bmatrix} \tag{5.7}$$

where $\hat{g}$ and $\hat{h}$ are continuous functions with absolutely convergent Fourier series, $\hat{g}$ is real nonnegative and even, $\hat{h}$ is purely imaginary and odd, and the matrix $\hat{f}(\theta)$ is positive semidefinite. We shall call such states admissible limiting states. The following result shows that they are obtained by a "Boltzmann–Gibbs" prescription. We denote here by $\mathcal{X}^0$ the set of finite oscillator configurations, which can be considered as a subset of $\mathcal{X}$.

Proposition 5.1. Let P be an admissible limiting state with spectral density $\hat{f}$ given by Eq. (5.4), and moreover such that $\det \hat{f}(\theta) \geqslant c > 0$ for all $\theta \in [-\pi, \pi]$. Then

(i) $$\mathbb{E}_P e_k = g_k, \qquad \mathbb{E}_P a_k = h_k, \quad \text{and}$$

(ii) P is a Gibbs state with potential $H(x) = \sum_{k\in\mathbb{Z}} \lambda_k \mathcal{E}_k(x) + \mu_k \mathcal{Q}_k(x)$, $x \in \mathcal{X}^0$ where λ_k and μ_k, $k \in \mathbb{Z}$ are the Fourier coefficients of the functions $\hat{f}_1(\theta) = 2\hat{g}(\theta)/[\hat{g}^2(\theta) + \omega^2(\theta)\hat{h}^2(\theta)]$ and $\hat{f}_2(\theta) = -2\hat{h}(\theta)\omega^2(\theta)/[\hat{g}^2(\theta) + \omega^2(\theta)\hat{h}^2(\theta)]$, respectively.

Proof. Observe that since $\hat{g}$, $\hat{h}$ have an absolutely convergent Fourier series, the same is true for the function $\hat{g}^2 + \omega^2\hat{h}^2$, and, since this function does not vanish by hypothesis, the functions $\hat{f}_1$ and $\hat{f}_2$ again have an absolutely summable Fourier series (see Ref. 16, Chap. II, Sec. 2). Therefore, by a simple extension of a result of Dobrushin[17] it is easy to see that a Gaussian state satisfying our assumptions is a Gibbs state: its potential $H(x)$ is a quadratic form given by the Fourier coefficients of the inverse matrix $\hat{f}^{-1}(\theta)$ (see the proof of Theorem 4.1 in Ref. 17). Here

$$\hat{f}^{-1}(\theta) = \frac{2}{\hat{g}^2(\theta) + \omega^2(\theta)\hat{h}^2(\theta)} \begin{bmatrix} \omega^2(\theta)\hat{g}(\theta) & -\hat{h}(\theta)\omega^2(\theta) \\ \hat{h}(\theta)\omega^2(\theta) & \hat{g}(\theta) \end{bmatrix}$$

and therefore

$$\begin{aligned} H(x) &= \frac{1}{2} \sum_{h,h'\in\mathbb{Z}} \left(x_h, (f^{-1})_{h-h'} x_{h'}\right) \\ &= \sum_{h,h'\in\mathbb{Z}} (\omega^2\hat{f}_1)_{h-h'} q_h q_{h'} + (\hat{f}_1)_{h-h'} p_h p_{h'} + (\hat{f}_2)_{h-h'} (q_h p_{h'} - p_h a_{h'}) \\ &= \sum_{k\in\mathbb{Z}} \{\lambda_k \mathcal{E}_k(x) + \mu_k \mathcal{Q}_k(x)\} \end{aligned}$$

which concludes the proof. ∎

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APPENDIX. ESTIMATES OF OSCILLATING INTEGRALS

We will prove here some results, mainly based on the method of stationary phase, which are used in the text.

Throughout the appendix $f, g : \mathbb{R}^1 \to \mathbb{R}^1$ will be 2π-periodic, at least locally summable functions; $\hat{h}$ will denote the function $\hat{h}(x,t) = g(x) \exp[itf(x)]$ and $h_k(t)$, $k \in \mathbb{Z}$, its Fourier coefficients.

Proposition A.1. If $f \in C^{k+1}$, $k \geqslant 2$ and there is no point $x \in [-\pi, \pi]$ for which $f'(x) = \cdots = f^k(x) = 0$, then

$$\lim_{t \to \infty} \int_{-\pi}^{\pi} \hat{h}(x,t)\, dx = 0$$

Proof. The hypotheses imply that f' has at most a finite number of zeros. [If it is not so and $\bar{x}$ is an accumulation point of the zeroes we would have $f'(\bar{x}) = \cdots = f^{(k)}(\bar{x}) = 0$.] Let $x_1, \ldots, x_n$ be the zeroes of f' and suppose that $g \in C^1$. We split the integral in a sum of integrals over small intervals I_i around x_i, $i = 1, \ldots, n$ plus a remainder, which, integrating by parts, is seen to be $O(|t|^{-1})$. Furthermore, performing a canonical change of variables (see Ref. 18, Lemma 1.1.4) we find $|\int_{I_i} \hat{h}(x,t)\, dx| = |\int_{\tilde{I}_i} g(\varphi(y)) \exp(ity^n)\, d\varphi(y)$, where $\varphi \in C^1$, $n \leqslant k$ and $\tilde{I}_i$ is an interval containing the origin; its length $|\tilde{I}_i|$ goes to 0 linearly with $|I_i|$. Since $g(\varphi)\varphi'$ is continuous, applying Erdelyi's lemma (Ref. 18, Lemma 3.1.2) we see that the integral over $\tilde{I}_i$ goes to 0 as $t \to \infty$. This proves the result for $g \in C^1$; a density argument gives the proof for $g \in L_1[-\pi, \pi]$. ■

Proposition A.2. If $f,g \in C^2$ and $\bar{\gamma} = \max_{x \in [-\pi,\pi]} |f'(x)|$, for any $\gamma > \bar{\gamma}$ we have

$$\lim_{t \to \infty} \sum_{\substack{k \in \mathbb{Z} \\ |k| \geqslant \gamma t}} |h_k(t)| = 0$$

Proof. Setting $a = t/k$, $S_a(x) = af(x) - x$ we have

$$h_k(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) \exp[ikS_a(x)]\, dx$$

Since for $|a| \leqslant 1 \backslash \gamma$, $|S_a'(x)| = |af'(x) - 1| \geqslant |\gamma - \bar{\gamma}|/\gamma$, integrating twice by parts we find $|h_k(t)| \leqslant c/(1 + k^2)$ for some constant c depending on f, g, and γ. Therefore

$$\sum_{\substack{k \in \mathbb{Z} \\ |k| \geqslant \gamma t}} |h_k(t)| \leqslant c \sum_{\substack{k \in \mathbb{Z} \\ |k| \geqslant \gamma t}} 1/(1 + k^2)$$

which proves the result. ■

Proposition A.3. Let $g \in C$, and $f \in C^3$ be such that the set of the points $x \in [-\pi, \pi]$ for which $f''(x) = f'''(x) = 0$, is empty. Then there is a constant C depending on f and g such that

$$|h_k(t)| \leqslant C/(|t|^{1/3} + 1)$$

Proof. As in Proposition A.1 above, it is easily seen that the null set of f'' is finite. Let $x_1, \ldots, x_n$, $x_1 < \cdots < x_n$ be these points, and let $\eta = \min_{i=1,\ldots,n}|f'''(x_i)|$. By hypothesis $\eta > 0$. Set $a(t) = |\eta t|^{-1/3}$, suppose $|t|$ is so large that

$$2a(t) < \min_{\substack{i,j=1,\ldots,n \\ i \neq j}} |x_i - x_j|$$

(distances are computed in the S^1 metric), and consider the intervals (in S^1)

$$I_i = \left[x_i - a(t), x_i + a(t)\right], \qquad i = 1, \ldots, n$$
$$J_i = (x_i + a(t), x_{i+1} - a(t)), \qquad i = 1, \ldots, n,$$
$$\left[J_n = (x_n + a(t), x_1 - a(t))\right]$$

For t large enough $\inf_{x \in J_i}|f''(x)|$ will be attained at the boundary of J_i, $i = 1, \ldots, n$. We have $\min_{i=1,\ldots,n}\inf_{x \in J_i}|f''(x)| \geqslant \eta a(t)$ and a van der Corput estimate (see Ref. 19) gives

$$\left|\int_{J_i} \hat{h}(x,t)\, dx\right| \leqslant c' \Big/ \left[\eta |t| a(t)\right]^{1/2}$$

for some constant c'. Since moreover $|\int_{I_i} \hat{h}(x,t)\, dx| \leqslant 2\|g\|_\infty a(t)$, we get for $|t|$ large enough $|h_k(t)| \leqslant c_1/|t|^{1/3}$, which proves the result. ■

Proposition A.4. Suppose that f satisfies the same assumptions of proposition A.1 and that $g \in C^1$. Then

(i) if $f'(0) \neq 0$,

$$\lim_{t \to +\infty} P\int_{-\pi}^{\pi} \frac{1}{x} \hat{h}(x,t)\, dx - \pi i h(0,t) \operatorname{sgn}\left[f'(0)\right] = 0$$

(ii) $P\int_{-\pi}^{\pi}(1/x)\hat{h}(x+y,t)\, dx$ is bounded uniformly for $y \in [-\pi, \pi]$, $t \in \mathbb{R}^1$.

Proof. We have

$$P\int_{-\pi}^{\pi} \frac{1}{x} \hat{h}(x,t)\, dx = \int_{-\pi}^{\pi} \frac{g(x) - g(0)}{x} \exp\left[if(x)t\right] dx$$
$$+ g(0) P\int_{-\pi}^{\pi} \frac{1}{x} \exp\left[if(x)t\right] dx$$

The first integral vanishes for $t \to \infty$ by Proposition A.1. If $f'(0) = a \neq 0$, we have for $|x| \leqslant \delta$, $f(x) - f(0) = ax[1 + r(x)]$ with $\max_{|x| \leqslant \delta} |r(x)| \leqslant c_1 \delta$. Set $z(x) = x[1 + r(x)]$. $z(x)$ is differentiable, $z(0) = 0$, and $z'(x) = 1 + s(x)$, with $\max_{|x| \leqslant \delta} |s(x)| \leqslant c_2 \delta$. Therefore if δ is small enough we can take z as a new variable, and, denoting by φ the inverse function we have

$$P \int_{-\delta}^{\delta} \frac{1}{x} \exp[if(x)t] \, dx = \exp[if(0)t] P \int_{-\delta}^{\delta} \frac{1}{z} \exp(iazt) \frac{z}{\varphi(z)} \varphi'(z) \, dz + o(1)$$

We have $[z/\varphi(z)]\varphi'(z) - 1 = z\chi(z)$, χ being a bounded function. Therefore

$$\lim_{t \to \infty} \left\{ P \int_{-\delta}^{\delta} \frac{1}{x} \exp[if(x)t] \, dx - \exp[if(0)t] P \int_{-\delta}^{\delta} \frac{1}{z} \exp(iatz) \, dz \right\} = 0$$

Since

$$\lim_{\lambda \to +\infty} \int_{-\delta}^{\delta} \frac{1}{x} \exp(\pm i\lambda x) \, dx = \pm \pi i$$

assertion (i) is proved.

To prove assertion (ii), without loss of generality we may assume $g = 1$. Consider the set $I_\delta = \{x \in S^1 : |x - x_i| \geqslant 2\delta, i = 1, \ldots, n\}$. For small δ, I_δ is a union of nonintersecting intervals. Let J be one of them. Clearly $|f'(x + y)| \geqslant \gamma > 0$ for $y \in J$, $|x| \leqslant \delta$, and $f(x + y) - f(x) = f'(y)x[1 + r_y(x)]$ where $r_y(x)$ is small for $|x| \leqslant \delta$, uniformly in $y \in J$. Taking $z = x[1 + r_y(x)]$ as a new variable and repeating the steps in the proof of assertion (i), it is easily seen that $P\int_{-\delta}^{\delta}(1/x)\exp[if(x + y)t]\,dx$ is uniformly bounded for $y \in J$. Hence assertion (ii) holds for $y \in I_\delta$. Suppose now that $|y - x_i| \leqslant 2\delta$ for some i, $i = 1, \ldots, n$. Without loss of generality we may assume $x_i = 0$. Suppose $f'(0) = \cdots = f^{(m-1)}(0) = 0$, $f^{(m)}(0) = a \neq 0$, for some m, $1 < m \leqslant k$. We can write $f(x) = f(0) + (a/m!)x^m[1 + r(x)]$, where r is a function of class C^1 at least and $r(0) = 0$. Take as a new variable the function

$$z = (x + y)[1 + r(x + y)]^{1/m} - \mu(y), \qquad \mu(y) = y[1 + r(y)]^{1/m}$$

Clearly $z(0) = 0$, z is differentiable in x and $z'(x) = 1 + s_y(x)$, with $|s_y(x)| \leqslant c_3\delta$ for $|x| \leqslant \delta$, $|y| < 2\delta$. Therefore for δ small enough, denoting by φ_y the inverse function of z, we find

$$P \int_{-\delta}^{\delta} \frac{\exp[if(x + y)t]}{x} \, dx = e^{if(0)t} P \int_{-\delta}^{\delta} \frac{\exp[i(a/m!)t(z + \mu(y))^m]}{z} \times \frac{z}{\varphi(z)} \varphi'(z) \, dz + o(1)$$

As above $\varphi_y'(z)[z/\varphi_y(z)] - 1 = z\chi_y(z)$, ψ_y being a function bounded uniformly in $|y| < 2\delta$, $|z| \leqslant \delta$. To conclude the proof we show that the integral

$$I(y,\lambda) = P\int_{-\delta}^{\delta} \frac{\exp\left[i\lambda(z+\mu(y))^m\right]}{z}\,dz$$

is bounded for all $\lambda \in \mathbb{R}^1$ and $y \in (-2\delta, 2\delta)$. Clearly if $|\lambda| \leqslant (2/\delta)^m$ it is bounded, since if $\varphi \in C^1$

$$\left| P\int_{-\delta}^{\delta} \frac{\varphi(x)}{x}\,dx \right| \leqslant 2\delta \max_{|x|\leqslant\delta} |\varphi'(x)|$$

Therefore let $|\lambda| > (2/\delta)^m$ and suppose for definiteness that $\lambda > 0$, $\mu > 0$. Set $\lambda^{1/m}\mu = \nu$ and consider the case $\nu \leqslant 1$. We have

$$\begin{aligned} |I(y,\lambda)| &= \left| P\int_{-\delta\lambda^{1/m}}^{\delta\lambda^{1/m}} \frac{e^{i(w+\nu)^m}}{w}\,dw \right| \\ &\leqslant \left| P\int_{-2}^{2} \frac{e^{i(w+\nu)^m}}{w}\,dw \right| + \left| \int_{2}^{\delta\lambda^{1/m}} \frac{e^{i(w+\nu)^m} - e^{i(-w+\nu)^m}}{w}\,dw \right| \end{aligned}$$

For $\nu \leqslant 1$ the first integral is bounded by inequality (A.1). The second integral is also bounded, as can be seen integrating by parts. If $\nu > 1$, by a change of variables we find, setting $\beta = \nu^m$ and $\delta' = \delta/\mu(2\delta) = \min_{0\leqslant y\leqslant 2\delta}\delta/\mu(y)$

$$\begin{aligned} |I(y,\lambda)| &= \left| P\int_{-\delta\mu^{-1}}^{\delta\mu^{-1}} e^{i\beta(u+1)^m}\frac{du}{u} \right| \\ &\leqslant \left| \int_{-\delta'}^{\delta\mu^{-1}} \frac{e^{i\beta(u+1)^m} - e^{i\beta(-u+1)^m}}{u}\,du \right| + \left| P\int_{-\delta'}^{\delta'} \frac{e^{i\beta(u+1)^m}}{u}\,du \right| \end{aligned}$$

Clearly $\delta' < 1$ for δ small enough. The second integral is bounded for all β by assertion (i). The first integral is bounded by $4/\delta'$ if $\delta\mu^{-1} \leqslant 2$. If $\delta\mu^{-1} > 2$ one must add to $4/\delta'$ an estimate of the integral from 2 to $\delta\mu^{-1}$, which is easily done integrating by parts. Similar considerations apply for the other choices of the signs of λ an μ. Proposition A.4 is proved. ■

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